Sec. 1.6 Algebraic properties of matrix operations

① Properties of addition

Same as for scalars. See Thm. 7.

② Properties of multiplication

Most (but not all — see topic ③) are similar to those for scalars.
Associative property holds:

$(AB)C = A(BC)$ Thm. 8 ← MUST READ

Distributive properties hold: see Thm. 9.

③ Differences from scalar multiplication.

(D-1) $AB \neq BA$ in general (see Ex. 4 in Sec. 1.5 (notes)).

(D-2) $(AB = AC) \overset{\text{in general}}{\not\Longrightarrow} B = C$,

even if $A \neq \mathcal{O}$ ← zero matrix.

Ex. 1 $A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 3 & 1 \\ -1 & 6 \end{pmatrix}$.

$AB = \begin{pmatrix} 10 & 16 \\ 5 & 8 \end{pmatrix} = AC$.

Corollary:

$$(PQ = \mathcal{O}) \overset{\text{in general}}{\not\Longrightarrow} (P \text{ or } Q = \mathcal{O}).$$

Proof: starting with (D-2):

$$(AB = AC) \not\Longrightarrow (B = C)$$

$$\downarrow \text{Thm. 7}$$

$$(AB - AC = \mathcal{O}) \not\Longrightarrow (B - C = \mathcal{O})$$

$$\downarrow \text{Thm. 9}$$

$$(\underbrace{A}_{P}\underbrace{(B-C)}_{Q} = \mathcal{O}) \not\Longrightarrow (\underbrace{B-C}_{Q} = \mathcal{O}),$$

and also $\underbrace{A}_{P} \neq \mathcal{O}$.

For an example with numbers, take $P = A$ and $Q = B - C$, with A, B, C in Ex. 1.

4 Scalar product and transpose of a matrix

Recall the dot product in Calculus:

$\vec{a} = \langle a_1, a_2 \rangle$, $\vec{b} = \langle b_1, b_2 \rangle$, $\Rightarrow \vec{a} \bullet \vec{b} = a_1 b_1 + a_2 b_2$.

Our notations: $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

Define the scalar product of $\underline{a}$ and $\underline{b}$:

$$\boxed{\underline{a}^T \underline{b} \equiv (a_1, a_2) \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2}$$

"transpose": make a column a row (and vice versa; see below).

In general, the scalar product of

$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $\underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ is:

$$\boxed{\underline{x}^T \underline{y} \equiv (x_1, x_2, \ldots x_n) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n}$$

"x-transpose".

<u>Transpose of a matrix A</u>: A^T.

To get A^T, make each column of A a row (and vice versa – this occurs automatically).

See p. 63 in book for a rigorous definition.

<u>Ex 2.</u> Find the transpose of:

(a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \end{pmatrix}$. $\Rightarrow A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 0 \end{pmatrix}$

2×3 $\qquad$ 3×2

(b) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. $\Rightarrow A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

(c) $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. $\Rightarrow A^T = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = A!$

<u>Def</u>: Matrices such that $A^T = A$ are called <u>symmetric</u>.

<u>Ex. 3</u> of a symmetric 3x3 matrix

$$A = \begin{pmatrix} 1 & 2 & \boxed{3} \\ 2 & 4 & 5 \\ \boxed{3} & 5 & 6 \end{pmatrix}$$

Thm. 10 Properties of the transpose.

1) Let A & B are $m \times n$. Then
$$(A+B)^T = A^T + B^T.$$

2) Let A be $m \times n$ and B be $n \times p$. Then
$$(AB)^T = B^T A^T \longleftarrow \boxed{\text{NOTE THE ORDER!}}$$

Why does one need to "flip" the order of A & B? Check the dimensions:

$$\underbrace{\underset{m\times n = n \times p}{A \quad B}}_{m \times p} \Rightarrow (AB)^T \text{ is } p \times m.$$

$$\underbrace{\underset{p\times n = n \times m}{\underset{(n\times p)^T \ (m\times n)^T}{B^T \quad A^T}}}_{p \times m} \Rightarrow B^T A^T \text{ is also } p \times m.$$

But $\underset{n\times m \,??\, p \times n}{A^T B^T}$ may be undefined!

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- Special case: if r is a scalar, then
$$(rA)^T = r A^T.$$

3) $(A^T)^T = A$.

Proofs are in the book.

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(5) **On your own:** ← MUST READ !

pp. 65-67 : about the identity matrix (analogue of "1" for scalars)

9/14/12 pp 67, 68 : about vector norm (= "length").

(6) Some examples of proofs

Ex. 4 Show that for any $m \times n$ matrix A, $B = A^TA$ and $C = AA^T$ are always symmetric.

Proof : 0) Make sure B & C are defined:

$\underset{(m\times n)^T \; m\times n}{B = A^TA}$ is $n \times n$; similarly C is $m \times m$. ✓

$n \times m = m \times n$

$n \times n$

Do for B, similar for C → 1) Have: $B = A^TA$ Want: $B^T = B$.

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So compute $\underbrace{B^T}_{\text{l.h.s.}}$ and compare with $\underbrace{B}_{\text{r.h.s.}}$.

$$B^T = (A^TA)^T \overset{2)}{=} A^T(A^T)^T \overset{3)}{=} A^TA = B.$$

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Ex. 5 Find A, B s.t. A & B are symmetric, but AB is not.

Sol'n:

Have:
$A^T = A$, $B^T = B$
$C = AB$

Want:
$C^T \neq C$.

So, compute C^T and compare it with C.

$$C^T = (AB)^T \overset{2)}{=} B^TA^T \overset{\text{given}}{=} BA$$
$$C = AB.$$

We know that $BA \neq AB$ in general, so it should be easy to find such A, B.

For example:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 5 & 0 \\ 4 & 3 \end{pmatrix} \leftarrow \text{not symmetric.}$$

HW: 7, 13, 15, 17, 19, 21, 26, 28, 27, 31 [30 was done in class], 35, 41(a), 43, 47, 49(b), 57.

EC #2: 46 + Sec. 1.5 #67.

Note: B is $n \times n$, not $m \times n$, as book says.

Hint for 41(a): write this as a l.s.

for 43(b): Use Thm. 8 repeatedly. First do the problem for $A^2\underline{u}$, then move on to $A^5\underline{u}$.