

Sec. 1.7. Linear independence & Nonsingular matrices.

PART A:

① Linear independence of vectors

NOTE: The material of both parts of this section will be FUNDAMENTAL for understanding most concepts developed in the rest of the course.

Def: $\underline{\theta} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is called the zero vector.

Def1: A set of vectors $\{v_1, v_2, \dots, v_p\}$ is called linearly independent

if the only sol'n to the vector eqn:

$$c_1 v_1 + c_2 v_2 + \dots + c_p v_p = \underline{\theta} \quad (\star)$$

is $c_1 = c_2 = \dots = c_p = 0$.

(A set of vectors is called linearly dependent if it is not lin. independent.)

Def*: A set $\{v_1, \dots, v_p\}$ is lin. dependent when (\star) has a sol'n where not all of c_1, c_2, \dots, c_p are 0.

Q: Why the name "linearly dependent"?

(Do this for 3 vectors $\underline{v}_1, \underline{v}_2, \underline{v}_3$.)

- Intuitively, we want to say that $\underline{v}_1, \underline{v}_2, \underline{v}_3$ are dependent when one of them depends on the other two, e.g.:

$$\underline{v}_3 = k_1 \underline{v}_1 + k_2 \underline{v}_2 \quad (\star)$$

for some k_1, k_2 .

How does this give with the Def.* above?

- Rewrite (\star) as:

$$\begin{array}{ccccc} k_1 \underline{v}_1 + k_2 \underline{v}_2 + (-1) \underline{v}_3 = \underline{0} \\ \uparrow \quad \quad \quad \uparrow \quad \quad \quad \uparrow \\ c_1 \quad \quad \quad c_2 \quad \quad \quad c_3 \end{array}$$

of c_1, c_2, c_3 , at least one ($c_3 = -1$) is not 0.

So, our intuitive (not mathematical!) definition is: ✓

(Set $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p\}$ is lin. dependent)
 \iff
 (one vector is a linear combination of the rest.)

Ex 1 Determine if set $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is lin. dependent or independent:

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}, \quad \underline{v}_3 = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}$$

Sol'n: 1) Setup the definition (\star) :

$$c_1 \underline{v}_1 + c_2 \underline{v}_2 + c_3 \underline{v}_3 = \underline{0} \quad (\star)$$

and proceed to solve for c_1, c_2, c_3 .

2) From (A), use the **KEY FORMULA** of Sec. 1.5 to write (A) as:

$$[\underline{v}_1, \underline{v}_2, \underline{v}_3] \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \underline{0}$$

This is a l.s. (written in matrix form), \Rightarrow solve it.

3) Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & -1 & 5 & 0 \\ 3 & 4 & 2 & 0 \end{array} \right) \xrightarrow{\text{REF}} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} c_1 + 2c_3 = 0 \\ c_2 - c_3 = 0 \\ c_3 = \text{free} \end{array}$$

$$\Rightarrow c_1 = -2c_3, c_2 = c_3, c_3 = \text{free.}$$

Since $c_3 \neq 0$ in general, we've found a sol'n of (A) with some nonzero c_1, c_2, c_3 , \Rightarrow by the definition, $\underline{v}_1, \underline{v}_2, \underline{v}_3$ are lin. dependent.

• Let's check that this answer agrees with our intuitive definition above.

$$c_1 \underline{v}_1 + c_2 \underline{v}_2 + c_3 \underline{v}_3 = \underline{0} \quad \Rightarrow \quad \frac{-2c_3 \underline{v}_1 + c_3 \underline{v}_2 + c_3 \underline{v}_3 = \underline{0}}{c_3 \text{ (since } c_3 \neq 0)}$$

$$\Rightarrow -2\underline{v}_1 + \underline{v}_2 + \underline{v}_3 = \underline{0} \quad \Rightarrow \quad \boxed{\underline{v}_3 = 2\underline{v}_1 - \underline{v}_2} \quad \checkmark$$

Ex. 2 The set of **unit vectors** is lin. independent:

$$\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \underline{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

See p. 75 in textbook.

Ex. 3 (This Example will be extensively used later)

For what value of a is set $\{v_1, v_2\}$ lin. independent, where $v_1 = \begin{pmatrix} 1 \\ a \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$?

Sol'n: (follow the lines of Ex. 1)

1) Setup: $c_1 v_1 + c_2 v_2 = \underline{0}$; seek c_1, c_2 .

2) l.s. in matrix form: $[v_1, v_2] \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \underline{0}$
(by the Key Formula)

3) Sol'n using augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ a & 3 & 0 \end{array} \right) \xrightarrow{R_2 - aR_1 \rightarrow R_2} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 3-2a & 0 \end{array} \right) \dots$$

Need to consider 2 cases:

• $3-2a \neq 0$ ($a \neq 3/2$)

\Rightarrow can divide by $(3-2a)$, \Rightarrow get $\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \end{array} \right) \xrightarrow{\text{REF}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right)$

$\Rightarrow c_1 = 0, c_2 = 0$ is the only sol'n of $(*)$, \Rightarrow vectors are lin. independent (for $a \neq 3/2$).

• $3-2a = 0$ ($a = 3/2$)

$\Rightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right) \Rightarrow c_1 + 2c_2 = 0 \Rightarrow c_1 = -2c_2$
 $c_2 = \text{free} \quad c_2 = \text{free}$,

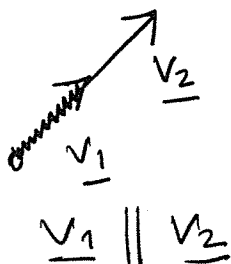
$\Rightarrow (*)$ has nonzero sol'ns ($c_1 = -2c_2 \neq 0$), \Rightarrow the vectors are lin. dependent for $a = 3/2$.

We have answered the formal question of the problem, but let's look at the answer

6-5

Geometrically:

$$a = 3/2 \quad \underline{v}_1 = \begin{pmatrix} 1 \\ 3/2 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow \underline{v}_2 = 2 \underline{v}_1$$



So: (2) vectors are lin. dep.
 \Leftrightarrow
(they are parallel)

Note: For (3) (or more) vectors,

the criterion of linear dependence

is different (they don't have to be \parallel ,
but still can be lin. dependent).

We will discuss this criterion next.

Ex. 4 W/o a calculation, show that

$\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \underline{v}_3 = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ are lin. dependent.

Sol'n: (begin by following Ex. 1)

1) $c_1 \underline{v}_1 + c_2 \underline{v}_2 + c_3 \underline{v}_3 = \underline{0}$ (setup to check linear dependence.)

2) Matrix form, by Key Formula:

$$[\underline{v}_1, \underline{v}_2, \underline{v}_3] \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \underline{0}$$

3) Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 2 & 4 & 6 & 0 \end{array} \right) \text{ This l.s. is:}$$

← Thm. 4 $\left\{ \begin{array}{l} \bullet \text{ homogeneous;} \\ \bullet 2 \times 3, 2 < 3 \end{array} \right.$

It has ∞ many sol'n's, \Rightarrow it has nonzero $c_{1,2,3}$.
Then $\underline{v}_1, \underline{v}_2, \underline{v}_3$ are lin. dep. by Def*.

So, in a plane, any 3 vectors are lin. dependent.

This can be generalized:

Thm. 11 let $\{\underline{v}_1, \dots, \underline{v}_n\}$ be a set of vectors in \mathbb{R}^m , and let $m < n$. Then these vectors are lin. dependent.

Clarification: This thm. does not apply when $m \geq n$. I.e., n vectors with $n \leq m$ may or may not be lin. dep.; there is no general conclusion.

E.g., let $m = 3, n = 2$.

- Take $\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$.

They are lin. dep.: $\underline{v}_2 = 2 \underline{v}_1$.

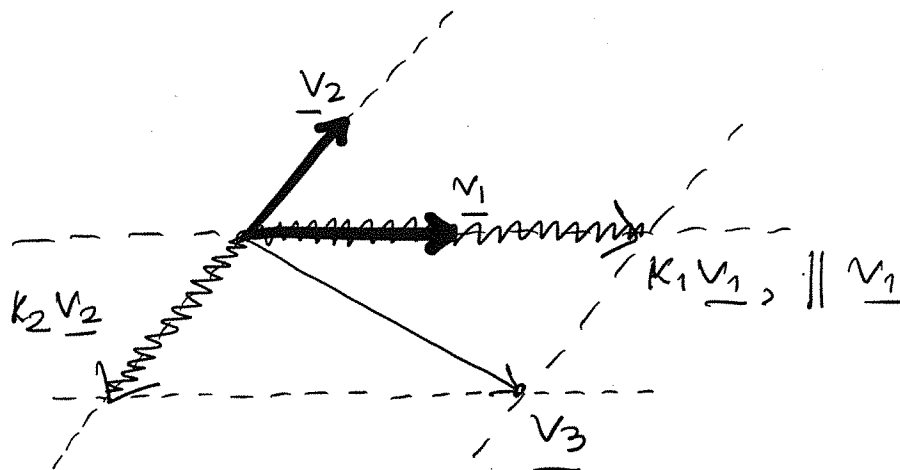
- But, take now $\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

These are clearly lin. indep.

Now, let's look again at 3 vectors in \mathbb{R}^2 .

Geometrically:

$$\underline{v}_3 = k_1 \underline{v}_1 + k_2 \underline{v}_2 \quad \text{means}$$



"parallelogram"

Practice this concept using the Matlab code posted after these Notes.

So, a geometric criterion of linear dependence of 3 vectors:

(3 vectors are lin. dependent)

\Leftrightarrow

(they lie in the same plane).

See the Summary of lin. dependence of 2 & 3 vectors on next page.

Summary of Ex. 3 & 4

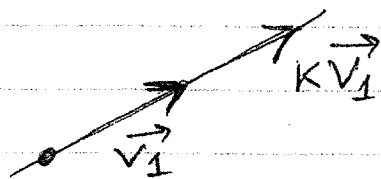
2D
and
3D

(a) If $\vec{v}_2 \parallel \vec{v}_1 \Rightarrow$
 \vec{v}_2 & \vec{v}_1 are lin. dependent.

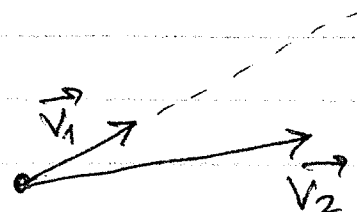
Indeed:

$$\vec{v}_2 = k \vec{v}_1$$

(\vec{v}_2 depends on \vec{v}_1).

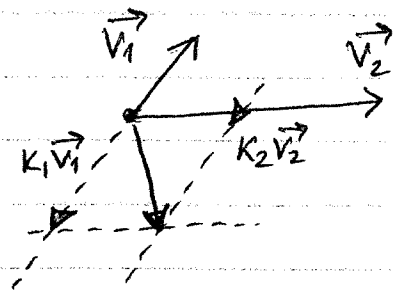


(b) If $\vec{v}_2 \nparallel \vec{v}_1 \Rightarrow$
 \vec{v}_2 & \vec{v}_1 are lin. independent:
there is no k s.t. $\vec{v}_2 = k \vec{v}_1$.



(c) Let $\vec{v}_1 \nparallel \vec{v}_2$.
Then any \vec{v}_3 in the plane
defined by \vec{v}_1, \vec{v}_2 is
lin. dependent on \vec{v}_1 & \vec{v}_2 :

$$\vec{v}_3 = k_1 \vec{v}_1 + k_2 \vec{v}_2$$



3D

(d) If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are not
in the same plane, then
they are lin. independent:

$$\vec{v}_3 \neq k_1 \vec{v}_1 + k_2 \vec{v}_2$$

for any k_1, k_2 .

