Sec. 1.7 – PART 2

2 Nonsingular & singular matrices

Def: An $n \times n$ matrix A is nonsingular if the only sol'n of $A\underline{x} = \underline{\theta}$ is $\underline{x} = \underline{\theta}$.

Equivalently, we have:

Def*: An $n \times n$ matrix A is singular if there is some $\underline{x} \neq \underline{\theta}$ such that $A\underline{x} = \underline{\theta}$.

$\boxed{\text{Note: Singular is Special.}}$

Relation with topic 1:

We can write $A = [\underline{A_1}, \underline{A_2}, \dots \underline{A_n}]$. Then:

$$A\underline{x} = \underline{\theta} \iff x_1\underline{A_1} + x_2\underline{A_2} + \dots + x_n\underline{A_n} = \underline{\theta}.$$

Also, $\underline{x} = \underline{\theta} \iff x_1 = x_2 = \dots = x_n = 0$.

Then the above Def's are equivalent to:

Thm. 12

$$\left(A \text{ is nonsingular}\right) \iff \left(\begin{matrix}\text{columns of } A\\ \text{are lin. independent}\end{matrix}\right)$$

$n \times n$

Thm. 12*

$$\left(A \text{ is singular}\right) \iff \left(\begin{matrix}\text{columns of } A\\ \text{are lin. dependent}\end{matrix}\right)$$

Ex. 5 For what a is matrix $A = \begin{pmatrix} 1 & 2 \\ a & 3 \end{pmatrix}$ nonsingular?

Sol'n: By Ex. 3, $\underline{A_1} = \begin{pmatrix} 1 \\ a \end{pmatrix}$ and $\underline{A_2} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ are lin. indep. for $a \neq 3/2$. Then by Thm. 12, A is nonsingular for $a \neq 3/2$.

Discussion: So, for $a=3/2$, A is singular. How does this conclusion jive with the definition of a singular matrix?

For $a=3/2$, $\underline{A}_1 = \begin{pmatrix} 1 \\ 3/2 \end{pmatrix}$, $\underline{A}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\Rightarrow \underline{A}_2 = 2\underline{A}_1$.

Rewrite this as:

$$2\underline{A}_1 - \underline{A}_2 = \underline{\theta} \quad \Leftrightarrow \quad 2\cdot\underline{A}_1 + (-1)\cdot\underline{A}_2 = \underline{\theta}$$

Key Formula $\Rightarrow$

$$\underbrace{[\underline{A}_1, \underline{A}_2]}_{A} \cdot \underbrace{\begin{pmatrix} 2 \\ -1 \end{pmatrix}}_{\underline{x}} = \underline{\theta}$$

So, the special $\underline{x}$ that makes $A\underline{x} = \underline{\theta}$ is $\underline{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ (for this particular A).

Ex. 6 Prove that the identity matrix (e.g. for 3×3, $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$) is nonsingular.

MUST + See Combined Ex. 6&1 posted under sec. 1.7B.

Discuss in class (emphasize that should **start with the Definition** of a sing. matrix, not lin. dep. of columns). Then computationally, it is the same proof as in our Ex. 2 (p. 75 in book). **see also Ex. 5 in book.**

Now we'll address the **BIG QUESTION**: When does an $n\times n$ l.s. $A\underline{x} = \underline{b}$ have a unique sol'n?

Thm. 13 Let A be $n\times n$.

$$\boxed{(A\underline{x} = \underline{b} \text{ has a unique sol'n}) \Leftrightarrow (A = \text{nonsingular})}$$

We'll only prove the "$\Leftarrow$", and only for 2×2 ($n\times n$ is similar)

$$(A = \text{nonsingular}) \Rightarrow \left(A\underline{x} = \underline{b} : \begin{array}{l} \text{(a) has a sol'n } \underline{x}_0; \\ \text{(b) this } \underline{x} \text{ is unique} \end{array}\right)$$

Proof.

Given:

- $\underline{A}\underline{x} = \underline{\theta}$ only for $\underline{x} = \underline{\theta}$
- $x_1 \underline{A}_1 + x_2 \underline{A}_2 = \underline{\theta}$ only for $x_1 = x_2 = 0$.

Want:

(a) Some $\underline{x}$ exists that satisfies $A\underline{x} = \underline{b}$.

(b) This $\underline{x}$ is unique.

(a) Consider a set

$\{\underline{A}_1, \underline{A}_2, \underline{b}\}$: 3 vectors in $\mathbb{R}^2$, $\Rightarrow$ by Thm. 11 or Ex. 4, they are lin. dependent.

Then $c_1 \underline{A}_1 + c_2 \underline{A}_2 + c_3 \underline{b} = \underline{\theta}$, $(\star)$

where some of c_1, c_2, c_3 must be $\neq 0$. $(\star\star)$

Now we argue that for sure, $c_3 \neq 0$. Why?

Because if $c_3 = 0$, then $(\star)$ reduces to

$c_1 \underline{A}_1 + c_2 \underline{A}_2 = \underline{\theta}$.

But by "Given", this is only possible for $c_1 = c_2 = 0$.

Then $c_3 = 0$ and $c_1 = c_2 = 0$, which contradicts $(\star\star)$.

So, $c_3 \neq 0$, and we can divide $(\star)$ by c_3:

$$\left(\frac{c_1}{c_3}\right)\underline{A}_1 + \left(\frac{c_2}{c_3}\right)\underline{A}_2 + \underline{b} = \underline{\theta}, \quad \Rightarrow$$

$$\underbrace{\left(-\frac{c_1}{c_3}\right)}_{\text{some } x_1}\underline{A}_1 + \underbrace{\left(-\frac{c_2}{c_3}\right)}_{\text{some } x_2}\underline{A}_2 = \underline{b}$$

$$x_1 \underline{A}_1 + x_2 \underline{A}_2 = \underline{b} \quad \Rightarrow \underbrace{[\underline{A}_1, \underline{A}_2]}_{A}\underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\text{some } \underline{x}} = \underline{b}$$

I.e., there indeed exists some $\underline{x}$ that satisfies $A\underline{x} = \underline{b}$.

(a) is proved

(b) Let's show that there are no other sol'ns.
By contradiction, suppose there are 2 sol'ns:
$A\underline{x} = \underline{b}$ and $A\underline{y} = \underline{b}$.
Subtract: $A\underline{x} - A\underline{y} = \underline{b} - \underline{b} \Rightarrow A\underbrace{(\underline{x}-\underline{y})}_{\underline{z}} = \underbrace{\underline{b}-\underline{b}}_{\underline{\theta}}$

$\Rightarrow A\underline{z} = \underline{\theta}$. But by "Given", this can be only when $\underline{z} = \underline{\theta} \Rightarrow \underline{x} - \underline{y} = \underline{\theta} \Rightarrow \underline{x} = \underline{y}$.

(b) is proved.

Proof of "$\Rightarrow$" in Thm. 13 (OPTIONAL):
($A\underline{x} = \underline{b}$ has a unique sol'n) $\Rightarrow$ (A = nonsingular).
Take $\underline{b} = \underline{\theta}$. Then $A\underline{x} = \underline{\theta}$ has a unique sol'n.
By inspection, $\underline{x} = \underline{\theta}$ is a sol'n, and now it is unique.
Then by Def of a nonsingular matrix, A = nonsingular.

Ex. 7. What can the number of solutions be for $\begin{pmatrix} 1 & 2 \\ a & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ c \end{pmatrix}$?

Sol'n: 1) Based on Thm. 13, we know that when the matrix is nonsingular, there is 1 sol'n. From Ex. 5, matrix is nonsingular for $a \neq 3/2$.
So, ($a \neq 3/2$) $\Rightarrow$ (1 sol'n; regardless of c).

2) What if $a = 3/2$? Then the matrix is singular. Use Thm. 13 to explore the possible number of sol'ns.

Thm. 13 says:

$$(A\underline{x}=\underline{b} \text{ has 1 sol'n}) \Rightarrow (A = \text{nonsingular}).$$

Negate this statement and use laws of logic:

$$(A=\text{singular}) \Rightarrow (A\underline{x}=\underline{b} \text{ has not 1 sol'n})$$

$\boxed{\text{means } 0 \text{ or } \infty}$

Now let's apply this to our example.

<u>$a=3/2$</u>

$$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 3/2 & 3 & c \end{array}\right) \xrightarrow[R_2-\frac{3}{2}R_1]{} \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & c-6 \end{array}\right)$$

- if $(c-6)\neq 0 \Rightarrow$ divide R_2 by $(c-6) \Rightarrow$

  $\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & 1 \end{array}\right) \Rightarrow$ the l.s. in inconsistent (Sec. 1.3)

  $\Rightarrow \boxed{0 \text{ sol'ns}}$

- if $c-6=0$ (i.e. $c=6$):

  $\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1+2x_2=4 \\ x_2 = \text{free} \end{array} \Rightarrow \boxed{\infty \text{ many sol'ns}}$

Thus, a <u>complement</u> to Thm. 13:

$$\boxed{(A=\text{singular}) \Rightarrow \left(\begin{array}{l} A\underline{x}=\underline{b} \text{ can have either} \\ 0 \text{ or } \infty \text{ many sol'ns} \end{array}\right)}$$

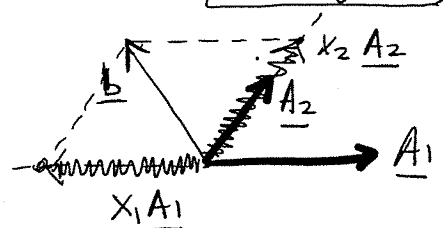
On the next page we illustrate this statement GEOMETRICALLY.

Geometric illustration of Thm. 13 and its complement:

number of sol'ns of $A\underline{x} = \underline{b}$.

$\underline{\mathbb{R}^2}$, i.e. $A = [\underline{A_1}, \underline{A_2}]$.

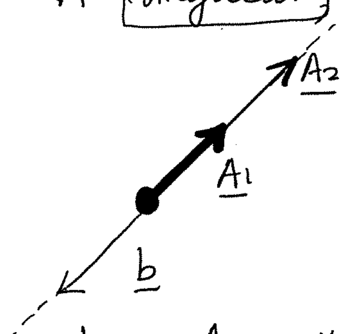
- A is $\boxed{\text{nonsingular}}$, $\underline{A_1}, \underline{A_2}$ lin. independent.



There is $\boxed{1 \text{ sol'n}}$ $\binom{x_1}{x_2}$ to

$$x_1 \underline{A_1} + x_2 \underline{A_2} = \underline{b}$$

- $A = \boxed{\text{singular}}$, $\underline{A_1}, \underline{A_2}$ are lin. dependent



$\underline{b} = x_1 A_1$ or $x_2 A_2$

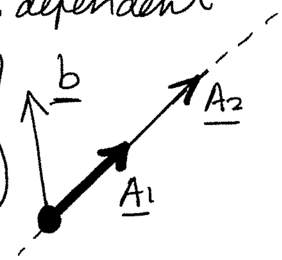
or

$x_1 \underline{A_1} + x_2 \underline{A_2}$

$\boxed{\infty \text{ many sol'ns}}$

or

not on the line made by $\underline{A_1}, \underline{A_2}$



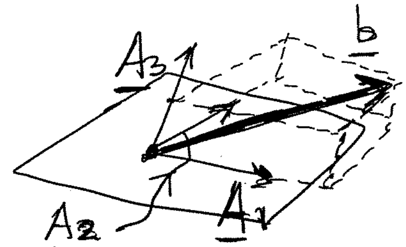
no x_1, x_2 will make

$x_1 \underline{A_1} + x_2 \underline{A_2} = \underline{b}$

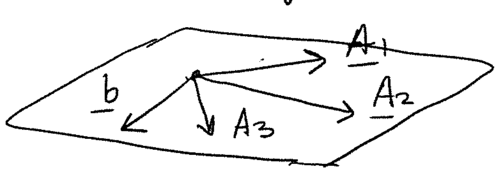
$\boxed{0 \text{ sol'ns}}$

$\underline{\mathbb{R}^3}$, i.e. $A = [\underline{A_1}, \underline{A_2}, \underline{A_3}]$

- A = nonsingular
  ($\underline{A_1}, \underline{A_2}, \underline{A_3}$ not in same plane)

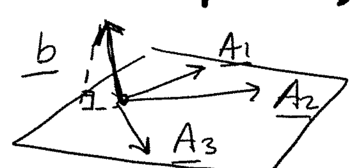


- A = singular ($\underline{A_1}, \underline{A_2}, \underline{A_3}$ **are in the same plane**)



$\underline{A_1}, \underline{A_2}, \underline{A_3}$ & $\underline{b}$ are all in same plane $\Rightarrow$ ∞ many sol'ns

or



$\underline{b}$ is not in the plane with $\underline{A_1}, \underline{A_2}, \underline{A_3}$, $\Rightarrow$ 0 sol'ns.

3 More examples of proofs.

Ex. 8 Recall that in Sec. 1.6 we had a strange property of matrices:

$$(PQ = \mathcal{O}) \underset{\text{in general}}{\not\Rightarrow} (P = \mathcal{O} \text{ or } Q = \mathcal{O})$$

"one cannot cancel by a nonzero matrix".

Now let's prove:

$$(PQ = \mathcal{O} \;\&\; P = \text{nonsingular}) \Rightarrow (Q = \mathcal{O}).$$

Observation:
- **one can "cancel" by a nonsingular matrix.**
- **one can cancel by a nonzero number**

"a nonsingular matrix is analogous to a nonzero number"

or

"a singular matrix is analogous to a zero number"

Proof: Given (for 2x2 matrices; $P \equiv [\underline{P}_1, \underline{P}_2]$ etc.)

$PQ = \mathcal{O}$

all these mean P = nonsingular:
- $P\underline{x} = \underline{\theta} \Rightarrow \underline{x} = \underline{\theta}$
- $x_1 \underline{P}_1 + x_2 \underline{P}_2 = \underline{\theta} \Rightarrow x_1 = x_2 = 0$
- $P\underline{x} = \underline{b}$ has 1 sol'n

(list all and then pick one that works; may need to use trial-and-error)

Want

$Q = \mathcal{O}$.

1) let $Q \equiv [\underline{Q}_1, \underline{Q}_2]$; then

$$PQ = \mathcal{O} \Rightarrow P[\underline{Q}_1, \underline{Q}_2] = \mathcal{O}$$

$$\Rightarrow [P\underline{Q}_1, P\underline{Q}_2] = [\underline{\theta}, \underline{\theta}] \Rightarrow \begin{cases} P\underline{Q}_1 = \underline{\theta} \\ P\underline{Q}_2 = \underline{\theta} \end{cases}$$

2) By "Given, 1st •", $\underline{P}\underline{Q}_1 = \underline{\theta} \Rightarrow \underline{Q}_1 = \underline{\theta}$.
Similarly, $\underline{Q}_2 = \underline{\theta}$. Then $Q = [\underline{Q}_1, \underline{Q}_2] = [\underline{\theta}, \underline{\theta}] = \mathcal{O}$.

Ex. 9 Let $A, B = n \times n$. Prove that:

(a) $(B = \text{singular}) \Rightarrow (AB = \text{singular})$

Proof: Given (see Def* on p. 6-9)

There is $\underline{x} \neq \underline{\theta}$ s.t. $B\underline{x} = \underline{\theta}$

this special $\underline{x} \neq \underline{\theta}$

Want:

There is some $\underline{y} \neq \underline{\theta}$ s.t. $(AB)\underline{y} = \underline{\theta}$.

Consider $A \cdot (B\underline{x} = \underline{\theta})$

$A(B\underline{x}) = A\underline{\theta} \Rightarrow (AB)\underline{x} = \underline{\theta}$

not $\underline{\theta}$.

Thus, we've found $\underline{y}\ (=\underline{x}) \neq \underline{\theta}$ s.t. $(AB)\underline{y} = \underline{\theta} \Rightarrow$ q.e.d.

(b) (OPTIONAL)

$(A = \text{singular}) \Rightarrow (AB = \text{singular})$

Proof: Given

There is $\underline{x} \neq \underline{\theta}$ s.t. $A\underline{x} = \underline{\theta}$

Want:

There is some $\underline{y} \neq \underline{\theta}$ s.t. $(AB)\underline{y} = \underline{\theta}$.

1) Case 1: $B = \text{singular}$.
Then $AB = \text{singular}$ by (a).

2) Case 2: $B = \text{nonsingular}$:
- Def. → $B\underline{z} = \underline{\theta} \Rightarrow \underline{z} = \underline{\theta}$
- Thm. 13 → $B\underline{z} = \underline{b}$ has 1 sol'n.

Take the $\underline{x} \neq \underline{\theta}$ from the "Given": $A\underline{x} = \underline{\theta}$

Consider a l.s. $\boxed{B\underline{z} = \underline{x}}$, where $\underline{x}$ is given, and $\underline{z}$ is to be found. By Thm. 13, such $\underline{z}$ exists (and is unique). Moreover, this $\underline{z} \neq \underline{\theta}$, because $B\cdot\underline{\theta} = \underline{\theta} \neq \underline{x}$.

Now:
$$A\cdot(B\underline{z} = \underline{x})$$
$$(AB)\underline{z} = A\underline{x} \to \underline{\theta} \text{ (given)}$$
$$(AB)\underline{z} = \underline{\theta}$$

$\underline{z} \neq \underline{\theta}$ (see above).

q.e.d.

Conclusion from Ex. 9(a,b):

$\boxed{(A \text{ or } B = \text{singular}) \Rightarrow (AB = \text{singular})}$

equivalently,

$\boxed{(AB = \text{nonsingular}) \Rightarrow (A \text{ and } B = \text{nonsingular})}$

Compare with the observation on p. 6-15:

$(a \text{ or } b = 0) \Rightarrow (ab = 0) \;\|\; (ab \neq 0) \Rightarrow (a \neq 0, b \neq 0)$

(4) Transpose & (non)singular

Claim: (proof = Extra Credit #3, #57)

$\boxed{(A = \text{nonsingular}) \Leftrightarrow (A^T = \text{nonsingular})}$

Corollary 1

$\boxed{(A = \text{singular}) \Leftrightarrow (A^T = \text{singular})}$

Corollary 2

$\left(\begin{array}{c}\textbf{Columns} \text{ of } A \\ \text{are lin. dep.}\end{array}\right) \Leftrightarrow \left(\begin{array}{c}\textbf{Rows} \text{ of } A \\ \text{are lin. dep.}\end{array}\right)$

<u>Proof</u> (OPTIONAL)

$$\begin{pmatrix}\text{Columns of } A\\ \text{are lin. dep.}\end{pmatrix} \underset{\text{Thm. 12*}}{\Longleftrightarrow} \begin{pmatrix} A = \text{sing.}\end{pmatrix} \underset{\text{Corollary 1}}{\Longleftrightarrow} \begin{pmatrix} A^T = \text{singular}\end{pmatrix}$$

$$\underset{\text{Thm. 12*}}{\Longleftrightarrow} \begin{pmatrix}\boxed{\text{Columns of } A^T}\\ \text{are lin. dep.}\end{pmatrix} \Longleftrightarrow \begin{pmatrix}\boxed{\text{Rows of } A}\\ \text{are lin. dep.}\end{pmatrix}.$$

same

Thus, if A = singular, <u>both</u> its columns <u>and</u> rows are lin. dependent.