Sec. 1.9

Matrix Inverse & Its Properties

1 Motivation

Analogy with scalars:

$$a \cdot \frac{1}{a} = 1, \quad \text{or} \quad a \cdot a^{-1} = 1$$

When do we usually use "division by a"?
When solving an equation!

$$ax = b \Rightarrow (\text{if } a \neq 0) \quad \frac{\cancel{a}x}{\cancel{a}} = \frac{b}{a}$$

$$\Rightarrow x = \frac{b}{a}.$$

Another way to write the same:

"scalar"

$$a^{-1} \cdot (ax = b) \Rightarrow \overset{1}{\cancel{(a^{-1}a)}} \cdot x = a^{-1}b \Rightarrow \boxed{x = a^{-1}b} \quad (S)$$

So, we want an analogue of formula (S) for l.s.

2 Definition and example of A^{-1}

Def: Let A be $n \times n$. Matrix B is called the inverse of A if:

$$\boxed{AB = I = BA.}$$

Notation: $B \equiv A^{-1}$ ("A inverse")

Ex. 1 Let $A = \begin{pmatrix} 1 & 2 \\ a & 3 \end{pmatrix}$. Verify that

$$A^{-1} = \frac{1}{3-2a} \cdot \begin{pmatrix} 3 & -2 \\ -a & 1 \end{pmatrix}.$$

Sol'n: It suffices to verify that $A \cdot A^{-1} = I$
(see the definition; $A^{-1} \cdot A = I$ is shown similarly).

So, $A \cdot A^{-1} = \begin{pmatrix} 1 & 2 \\ a & 3 \end{pmatrix} \cdot \frac{1}{3-2a} \cdot \begin{pmatrix} 3 & -2 \\ -a & 1 \end{pmatrix} =$

$= \frac{1}{3-2a} \left(\begin{array}{c|c} 1\cdot 3 + 2\cdot(-a) & 1\cdot(-2) + 2\cdot 1 \\ a\cdot 3 + 3\cdot(-a) & a\cdot(-2) + 3\cdot 1 \end{array} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

Note 1 : You may notice that entries of A^{-1} have some visible relation to entries of A. This is indeed so, but this is true only for 2×2 matrices. See pp. 98-99 of textbook for the formula (optional).

Note 2 Notice that A^{-1} does not exist when $3-2a=0$ (we have then $A^{-1} = \frac{1}{0}\cdot(\dots)$). But, $3-2a=0$, or $a=3/2$, is when A is singular (see Ex. 5 in Sec. 1.7/notes). So, we may suspect that the existence of A^{-1} and A being (non)singular are somehow related.

This agrees with the observation made in Sec. 1.7, that a singular A is the counterpart of the scalar 0, and we know that 0^{-1} does not exist. We'll soon confirm the suspicion stated above.

3 Usage of A^{-1}

We'll now follow the steps of the derivation of formula (S) on p. 8-1, which was obtained for a scalar equation, and apply it to a l.s. $A\underline{x} = \underline{b}$.

Let's assume that A^{-1} exists, and multiply by it:

$$A^{-1} \cdot (A\underline{x} = \underline{b})$$

$$A^{-1}(A\underline{x}) = A^{-1}\underline{b}$$

$$\underbrace{(A^{-1}A)}_{I}\underline{x} = A^{-1}\underline{b}$$

$$I\underline{x} = A^{-1}\underline{b}$$

$$\boxed{\underline{x} = A^{-1}\underline{b}} \qquad (V)$$

"vector"

This is the counterpart of (S) that we've sought.

Note 1 You can multiply by A^{-1} (if it exists), but **you can never divide by A!**

Note 2 Note the multiplication order: we were multiplying **on the left**. We cannot multiply our l.s. on the right:

$$(A\underline{x} = \underline{b}) \cdot A^{-1}$$

$$2\times 1 \qquad 2\times 2$$

dimensions do not match!

Even if we had overlooked the dimensions problem, we would have gotten on the l.h.s.:

$$A\underline{x}A^{-1} \neq AA^{-1}\underline{x}, \text{ because } \underline{x}A^{-1} \neq A^{-1}\underline{x}.$$

Ex. 2 Use the matrix inverse to solve the l.s.:

$$x_1 + 2x_2 = 4$$

$$x_1 + 3x_2 = 5.$$

Sol'n: 1) Put this l.s. in matrix form:

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\underline{x}} = \underbrace{\begin{pmatrix} 4 \\ 5 \end{pmatrix}}_{\underline{b}}$$

2) By formula (V) from the previous page:

$$\underline{x} = A^{-1} \underline{b}.$$

To find A^{-1}, use the formula from Ex. 1, because $\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ a & 3 \end{pmatrix}\Big|_{a=1}$. Then: $A^{-1} = \frac{1}{\underbrace{3-2\cdot 1}_{1}} \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$

3) Compute:

$$\underline{x} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

4) Existence and uniqueness of the inverse

On p. 8-2 we noted that there may be a link between A having the inverse and A being (non)singular. We now can state the precise relation.

Thm. 15 $\quad (A^{-1} \text{ exists}) \Leftrightarrow (A = \text{nonsingular})$

a) Proof of "$\Rightarrow$".

Given: A^{-1} exists

Want:
- $A\underline{x} = \underline{\theta} \Rightarrow \underline{x} = \underline{\theta}$ (Def)
- Columns of A are l. indep. (Thm. 12)
- $A\underline{x} = \underline{b}$ has 1 sol'n (Thm. 13)

So: $(A^{-1} \text{ exists}) \underset{\substack{\uparrow \\ \text{by formula (V) on p. 8-3}}}{\Rightarrow} (A\underline{x} = \underline{b} \text{ has 1 sol'n}) \Rightarrow \begin{pmatrix} A = \\ \text{nonsing.} \end{pmatrix}$ ✓

b) Proof of "$\Leftarrow$".

Instead of a usual proof, we'll work out an example that shows how A^{-1} can be constructed. We'll then comment where the condition (A = nonsing.) was used.

Ex. 3 Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$. Find A^{-1}.

Sol'n: This $A^{-1} = B$, where, by the Def. in topic ②, B satisfies $AB = I$.

Following the method of Ex. 6/Sec. 1.5, Ex. 8/Sec. 1.7: let $B \equiv [\underline{B}_1, \underline{B}_2]$. Then:

$$AB = I \Rightarrow A[\underline{B}_1, \underline{B}_2] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow [A\underline{B}_1, A\underline{B}_2] = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$$

- $A\underline{B}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 3 & 0 \end{array}\right) \xrightarrow[R_2 - R_1 \to R_2]{} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -1 \end{array}\right) \xrightarrow[R_1 - 2R_2 \to R_1]{} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right) \quad \underline{B}_1$$

- $A\underline{B}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 3 & 1 \end{array}\right) \xrightarrow[R_2 - R_1 \to R_2]{} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 1 \end{array}\right) \xrightarrow[R_1 - 2R_2 \to R_1]{} \left(\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 1 \end{array}\right) \quad \underline{B}_2$$

$$\Rightarrow B = [\underline{B}_1, \underline{B}_2] = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \leftarrow \text{Same as in Ex. 2.}$$

Note 1 The condition "A = nonsing." was used when we solved to l.s. $A\underline{B}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ etc.. By Thm. 13, the fact that A = nonsing. guaranteed that this l.s. had a (unique) sol'n, $\Rightarrow$ it guaranteed that we could find a unique B ($\equiv A^{-1}$).

<u>Note 2</u> The steps of the REF work for $\underline{B_1}$ & $\underline{B_2}$ were <u>exactly</u> the same. Thus, we can combine them:

$$\left[A \,\Big|\, \underbrace{\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)}_{I}\right] \xrightarrow[\text{REF}]{} \left[\begin{matrix}1 & 0\\0 & 1\end{matrix} \,\Big|\, \underbrace{(\underline{B_1}, \underline{B_2})}_{A^{-1}}\right]$$

Thus, the general algorithm is:

$$\boxed{[A \,|\, I] \xrightarrow[\text{REF}]{} [I \,|\, A^{-1}]}$$

See Ex. 3(a) and Ex. 2 in book for numeric examples.

<u>Note 3</u> If you are given some A and follow the above algorithm, but cannot get the "I" on the right-hand side? Then:

- either A = singular, or
- you've made a mistake.

This can happen in a HW problem, but <u>cannot</u> happen on a quiz or test!

<u>Note 4</u> (OPTIONAL)

In the Def., it is said that A and A^{-1} <u>commute</u>, which in general is not true for matrices, as we know since Sec. 1.5. So, <u>let's show now</u> that:

$$(AB = I) \Rightarrow (BA = I).$$

(The converse can be estabished similarly.)

1) Let's first show that B = nonsingular.

This follows from Ex. 9 in Sec. 1.7:

$$(AB (= I) = \text{nonsing.}) \Rightarrow (A \text{ and } B = \text{nonsing.})$$

2) Since B = nonsing., B^{-1} exists (by Thm. 15). Let's denote $B^{-1} = C$. Then, by Thm. 15, this C satisfies $BC = I$. Since we want to show that $BA = I$, it remains to show that $C = A$.

3) The calculation:

$$(AB = I)\cdot C \Rightarrow (AB)C = IC$$

$$\Rightarrow A(BC) = C \overset{2)}{\Rightarrow} A \cdot I = C \Rightarrow A = C.$$

q.e.d.

(5) Algebraic properties of the matrix inverse

Thm. 17 Let $A, B = n \times n$ and let A^{-1}, B^{-1} exist. Then:

1) $(A^{-1})^{-1} = A$ (Basically, we proved this in Note 4.)
2) $(AB)^{-1} = B^{-1}A^{-1}$ **(note the opposite order!)**
3) If k = scalar $\neq 0$, $\Rightarrow (kA)^{-1} = \frac{1}{k} A^{-1}$
4) $(A^T)^{-1} = (A^{-1})^T$

Proof of 2): It suffices to show that $(AB)\cdot(B^{-1}A^{-1}) = I$, because this means precisely that $B^{-1}A^{-1} = (AB)^{-1}$ by the Def. in topic (2).

So: $(AB)\cdot B^{-1}A^{-1} = A(BB^{-1})A^{-1} = A\cdot I \cdot A^{-1}$

$= A \cdot A^{-1} = I.$ ✓.

Q: Why wouldn't $A^{-1}B^{-1}$ work? Try it:

$$(AB)(A^{-1}B^{-1}) = A\underbrace{(BA^{-1})}_{\neq A^{-1}B}B^{-1} \Rightarrow$$

cannot combine A & A^{-1} and B & B^{-1}, $\Rightarrow$ stuck...

<u>Proof of 4)</u> (OPTIONAL)

We need to show that $(A^T)^{-1} = (A^{-1})^T$,

i.e. (by the Def. in topic 2), we want:

$A^T \cdot (A^{-1})^T = I$. Compute the l.h.s.:

$$A^T \cdot (A^{-1})^T \underset{\text{Thm. 10}}{=} (A^{-1} \cdot A)^T = I^T = I. \checkmark$$

<u>Thm. 18</u> (Summary of main results of Secs. 1.7, 1.9)

Let A be n×n. The following statements are equivalent:

1. A = nonsingular (Def. in 1.7-B)
2. Columns of A are lin. indep. (Thm. 12)
3. $A\underline{x} = \underline{b}$ has 1 sol'n (Thm. 13)
4. A^{-1} exists. (Thm. 15)
5. The REF of A is I. (used only in computation, not for proofs)

6 <u>More examples of proofs</u>

<u>Ex. 4</u>. In Sec. 1.6 we encountered a strange property of matrices:

$$\begin{pmatrix} AB = AC \\ \text{and} \\ A \neq \mathcal{O} \end{pmatrix} \underset{\text{in general}}{\not\Rightarrow} (B = C).$$

We can now modify this stament to have "⇒" hold:

$$\begin{pmatrix} AB = AC \\ \text{and} \\ A = \text{nonsing.} \end{pmatrix} \Rightarrow (B = C).$$

Proof: Given:
1) $AB = AC$
2) One of the 5 staments in Thm. 18 (The Summary)

Want:
$B = C$

We need to decide which statement in Thm. 18 to use.
We are in the section on matrix inverses ⇒ the common sense suggests that we use "A^{-1} exists".

Then: $AB = AC \Rightarrow A^{-1} \cdot (AB = AC)$
$$\Rightarrow (A^{-1}A)B = (A^{-1}A)C \Rightarrow I \cdot B = I \cdot C$$
$$\Rightarrow B = C. \checkmark$$

Note: Let us repeat the point made in Note 2 on p. 8-3: we must **multiply by A^{-1} on the left**.
Indeed, multiplication on the right would not work:
$$(AB)A^{-1} = (AC)A^{-1} \Rightarrow A(BA^{-1}) = A(CA^{-1})$$
$$\not\Rightarrow AA^{-1}B = AA^{-1}C, \text{ because}$$
$$BA^{-1} \neq A^{-1}B \text{ and } CA^{-1} \neq A^{-1}C.$$

Ex. 5. In Ex. 9 in Sec. 1.7 we showed
$$\boxed{(A \text{ or } B = \text{singular}) \Rightarrow (AB = \text{singular})} \quad (1a)$$
or, equivalently (by negation),
$$\boxed{(A \text{ and } B = \text{nonsingular}) \Leftarrow (AB = \text{nonsingular})} \quad (1b)$$

Let's now prove the converse of (1b):
$$(A \text{ and } B = \text{nonsingular}) \Rightarrow (AB = \text{nonsingular}) \quad (2b)$$
((2a) is not written; it is meant to be converse of (1a).)

<u>Proof</u>: Again, we are in the section about the matrix inverse, so it is the common sense to use statement 4 of Thm. 18:

$$(A \,\&\, B = \text{nonsing.}) \overset{\text{Thm. 18}}{\Longrightarrow} (A^{-1} \,\&\, B^{-1} \text{ exist})$$

$$\overset{\text{Thm. 17-2}}{\Longrightarrow} ((AB)^{-1} \text{ exists}) \underset{\text{or Thm. 18-4}}{\overset{\text{Thm. 15,}}{\Longrightarrow}} (AB = \text{nonsing.})$$

q.e.d.

So, combining (2b) & (1b):

$$\boxed{(A \,\&\, B = \text{nonsing.}) \Longleftrightarrow (AB = \text{nonsing.})} \quad (2)$$

by negation:

$$\boxed{(A \text{ or } B = \text{sing.}) \Longleftrightarrow (AB = \text{sing.})} \,. \quad (1)$$

This should make intuitive sense if we recall the earlier-stated analogy:
(a singular matrix) is the analogue of (number 0)
Then (1) is the matrix analogue of the statement:

$$(\text{numbers } a \,\&\, b \neq 0) \Longleftrightarrow (ab \neq 0).$$