Sec. 3.1 Review of Calc. II: Equations of lines & planes in $\mathbb{R}^2$ and $\mathbb{R}^3$.

In this Chapter we'll study so-called "vector spaces and subspaces" in $\mathbb{R}^n$ and their properties. We'll introduce a definition of a vector space later. For now, it will suffice to know that $\mathbb{R}^n$ is the set of all vectors with n real components.

Notation:

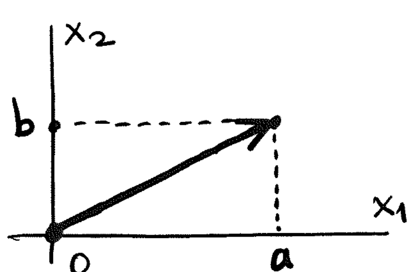
$$\mathbb{R}^2 = \{ \underline{x} : \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \; x_1, x_2 = \text{real} \}$$

"all $\underline{x}$" "such that" condition satisfied by $\underline{x}$

Why we care about the special cases of $\mathbb{R}^2$ & $\mathbb{R}^3$?

Most properties of $\mathbb{R}^n$ can be understood using the properties of lines in $\mathbb{R}^2$ and lines & planes in $\mathbb{R}^3$. We cannot visualize vectors in $\mathbb{R}^n$ for $n>3$, but we can visualize vectors, lines and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$.

① Eq. of a line in $\mathbb{R}^2$ which goes through $(0,0)$ along vector $\langle a, b \rangle$.



$$\boxed{\begin{aligned} x_1 &= at \\ x_2 &= bt \end{aligned}}$$

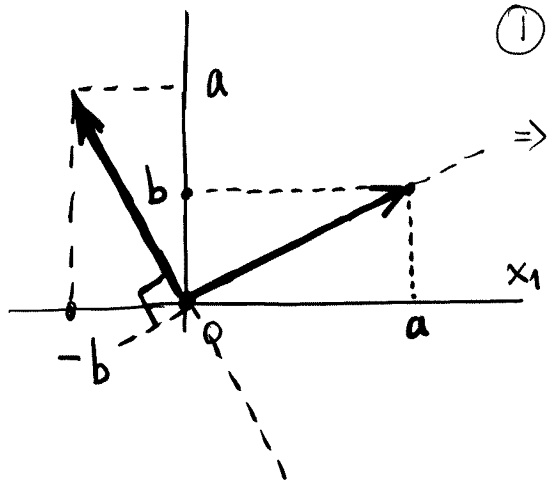
(Usual notations: $x = at$, $y = b \cdot t$)

Check: $y = b \cdot t = b \cdot \frac{x}{a} = \left(\frac{b}{a}\right) \cdot x$ ✓

$\left(\frac{b}{a}\right)$ ← slope

Notation: $\boxed{\{ \underline{x} : \underline{x} = \begin{pmatrix} at \\ bt \end{pmatrix}, t = \text{any real} \}}$

2 <u>Alternative form of 1</u>



1 $\Rightarrow$ $x_1 = at$, $x_2 = bt$

$\Rightarrow$ $-b\cdot(at) + a\cdot(bt) = 0$

$$\underbrace{-b}_{\substack{\uparrow \\ \text{call this} \\ A}} x_1 + \underbrace{a}_{\substack{\uparrow \\ \text{call this} \\ B}}\cdot x_2 = 0$$

$$\boxed{A\cdot x_1 + Bx_2 = 0}$$

> Memorize: An object is a line in $\mathbb{R}^2$ going through $(0,0)$ if and only if its equation has either of the forms given in 1 and 2.

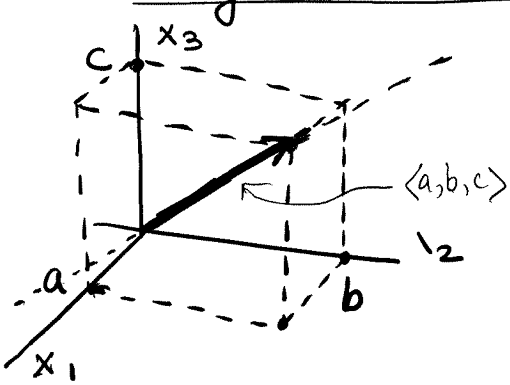
Note: $Ax_1 + Bx_2 = 0 \iff \begin{pmatrix}A\\B\end{pmatrix}^T\cdot\begin{pmatrix}x_1\\x_2\end{pmatrix} = 0$

$$\iff \underbrace{\begin{pmatrix}A\\B\end{pmatrix}}_{\begin{pmatrix}-b\\a\end{pmatrix}} \perp \underbrace{\begin{pmatrix}x_1\\x_2\end{pmatrix}}_{\parallel\begin{pmatrix}a\\b\end{pmatrix}}$$ ← this is what we see in the sketch above

3 <u>Eq. of a line in $\mathbb{R}^2$ not through origin</u>

$$Ax_1 + Bx_2 = C, \quad C \neq 0.$$

4 <u>Eq. of a line in $\mathbb{R}^3$ going through origin along vector $\langle a, b, c\rangle$.</u>



$$\boxed{\left\{\underline{x} : \underline{x} = \begin{pmatrix}at\\bt\\ct\end{pmatrix},\ t = \text{any real}\right\}}$$

Note: Eq. of a line in $\mathbb{R}^3$ **cannot** be written as $Ax_1 + Bx_2 + Cx_3 = 0$

In fact, to define a **line in $\mathbb{R}^3$**, we **need two eqs., not one**! See Ex. 2 below.

⑤ Eq. of a plane in $\mathbb{R}^3$ that is perpendicular to vector $\langle A, B, C \rangle$:

$$\boxed{A x_1 + B x_2 + C x_3 = D \qquad (\star)} \quad !$$

If $D = 0 \Rightarrow$ the plane goes through the origin.

Ex. 1 What is $\left\{ \underline{x} : \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},\ x_2 = x_1 \right\}$?

Sol'n: We see that this is in $\mathbb{R}^3$, so it can be either line (④) or plane (⑤). It does not appear like the eq. in ④. So it must be ⑤. Indeed, put eq. $x_2 = x_1$ in the form $(\star)$:

$$\underset{\substack{\uparrow \\ A}}{1} \cdot x_1 + \underset{\substack{\uparrow \\ B}}{\underbrace{(-1)}} \cdot x_2 + \underset{\substack{\uparrow \\ C}}{0} \cdot x_3 = \underset{\substack{\uparrow \\ D}}{0}$$

So, this is a plane $\perp$ vector $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, and since $D = 0$, $\Rightarrow$ this plane goes through the origin.

Geometric interpretation: count "degrees of freedom"

$$\underset{\substack{\text{(in } \mathbb{R}^3 \text{ one has} \\ \text{3 DoF)}}}{3} \quad - \quad \underset{\substack{\text{(1 constaining} \\ \text{equation)}}}{1} \quad = \quad \underset{\substack{\text{(\# of remaining} \\ \text{DoF)}}}{2}$$

A plane has 2 DoF, but a line has only 1 DoF.

Ex. 2 What is $\left\{ \underline{x} : \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, x_1 = x_2 \text{ and } x_2 = -x_3 \right\}$?

Sol'n: Write each constraining eq. in form (✱):

Eq. 1 $\Rightarrow$ $1 \cdot x_1 + (-1)\, x_2 + 0 \cdot x_3 = 0$ ← a plane through (0,0,0)

Eq. 2 $\Rightarrow$ $0 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 = 0$ ← another, non-parallel plane through (0,0,0)

The "and" says that $\underline{x}$ belongs to both planes.

Therefore, it is the intersection line of these planes, and goes through origin because each plane goes through origin.

Thus, answer: this is a line in R^3.

$\boxed{\text{Note: A line in } \mathbb{R}^3 \text{ is defined by two, not one, eq.}}$

Geometric interpretation: count "degrees of freedom"

$$\underset{(3 \text{ DoF in } \mathbb{R}^3)}{3} \quad - \quad \underset{(2 \text{ constraining eqs.})}{2} \quad = \quad \underset{(\# \text{ of remaining DoF})}{1}$$

1 DoF $\Rightarrow$ a line.