Sec. 3.2: Vector space properties of R^n

① Properties of R^n

Thm. 1 If $\underline{x}, \underline{y}$ are any vectors in R^n and c is any scalar, then the following groups of properties hold:

(A) ← addition of vectors
(M) ← multiplication by a scalar

} similar to corresponding properties of scalars, but SEE p. 168 in book.

(C) ← closure:

(C1) $(\underline{x}, \underline{y} \text{ in } R^n) \Rightarrow (\underline{x} + \underline{y} \text{ in } R^n)$

(C2) $(\underline{x} \text{ in } R^n) \Rightarrow (c \cdot \underline{x} \text{ in } R^n)$

Def: Any set of quantities that satisfies properties (A), (M), (C) is called a **vector space**.

This Def. & Thm. 1 imply:

Any set of quantities which satisfies (A), (M), (C) "behaves" exactly as vectors in R^n. (And, since vectors in R^n behave as vectors in R^2 and R^3, one can visualize any vector space by visualizing what goes on in R^2 and R^3.)

Here are two examples of vector spaces:

Ex. 1 $P = \{ p_n(x) \}$,

the set of all polynomials of x of degree $\le n$.

(Indeed, properties (A) & (M) are verified straightforwardly (once you look at p.168); then it remains to check the closure properties.

(C1): adding 2 polynomials of degree $\le n$, you always get a polynomial of degree $\le n$.

(C2): multiplying a polynomial of degree $\le n$ by a scalar (say, 3), you get a polynomial of degree $\le n$.)

<u>Ex. 2</u> The set of all <u>continuous functions</u> is a vector space.

(Indeed, the sum of two continuous functions is a continuous function (C1), and if you multiply a continuous function by scalar c, you get a continuous function.)

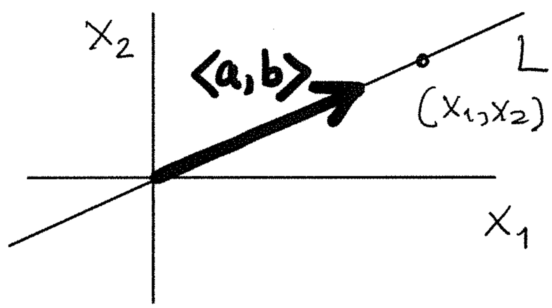
<u>Note:</u> In Project 2 you see an <u>explicit</u> relation between continuous functions $f(x)$ and vectors in $\mathbb{R}^n$: You <u>approximate</u> the elevation of a cable (a continuous function) by an <u>ordered</u> set of its values $(y_1, y_2, \dots y_n)$ at discrete points $(x_1, x_2, \dots)$.

- Yet another example of a vector space is the set of solutions of a <u>linear homogeneous</u> differential equation (**MATH 271**), and there are many more ....

Let us now look at two examples of subsets of $\mathbb{R}^2$, one of which is a vector space, and the other is not a vector space.

Ex. 3 A line passing through the origin in $\mathbb{R}^2$ is a vector space.

Method 1 – algebraic (it is still a good idea to make a sketch)



The fact that point (x_1, x_2) is on L is equivalent to the fact that vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\begin{pmatrix} a \\ b \end{pmatrix}$ are $\parallel$:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \parallel \begin{pmatrix} a \\ b \end{pmatrix} \underset{\text{topic ① in Sec. 3.1}}{\Longrightarrow} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \cdot t$$

for some t.

Then:

1) Properties (A) & (M) hold for all vectors (but you still need to look at p. 168!), and so **you will never need to check (A) & (M).**

So it remains to check (C1) & (C2).

2) Checking (C1): $(\underline{x}, \underline{y} \text{ in } L) \overset{?}{\Longrightarrow} (\underline{x}+\underline{y} \text{ in } L)$.

$\underline{x}$ in $L \Rightarrow \underline{x} = \begin{pmatrix} a \\ b \end{pmatrix} \cdot t_1$ for some t_1

$\underline{y}$ in $L \Rightarrow \underline{y} = \begin{pmatrix} a \\ b \end{pmatrix} t_2$ for some t_2

So, is $(\underline{x}+\underline{y}) = \begin{pmatrix} a \\ b \end{pmatrix} \cdot t_3$ for some t_3?

Check this: $\underline{x} + \underline{y} = \begin{pmatrix} a \\ b \end{pmatrix} t_1 + \begin{pmatrix} a \\ b \end{pmatrix} t_2 =$

$= \begin{pmatrix} a \\ b \end{pmatrix} \cdot \underbrace{(t_1 + t_2)}_{\text{call this "}t_3\text{"}}$

here we have used some properties from p. 168 ...

So, yes, indeed, (C1) holds. ✓

3) Checking (C2): $(\underline{x} \text{ in } L) \overset{?}{\Longrightarrow} (c \cdot \underline{x} \text{ in } L)$

$\underline{x}$ in $L \Rightarrow \underline{x} = \begin{pmatrix} a \\ b \end{pmatrix} t_1$ for some t_1.

So, is $c \cdot \underline{x} = \begin{pmatrix} a \\ b \end{pmatrix} \cdot t_4$ for some t_4 ?

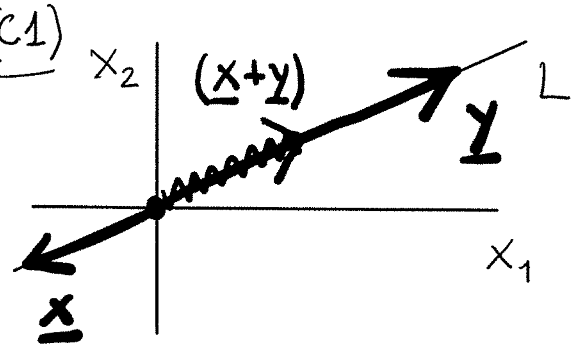
Check this: $c\underline{x} = c \cdot \begin{pmatrix} a \\ b \end{pmatrix} \cdot t_1 = \begin{pmatrix} a \\ b \end{pmatrix} \cdot \underbrace{(ct_1)}_{\text{call this "}t_4\text{"}}$.

So, yes, indeed, (C2) holds. ✓

Method 2 – geometric

We still need to check the same two properties (C1) & (C2), but now we do so with a sketch.

1) (C1)

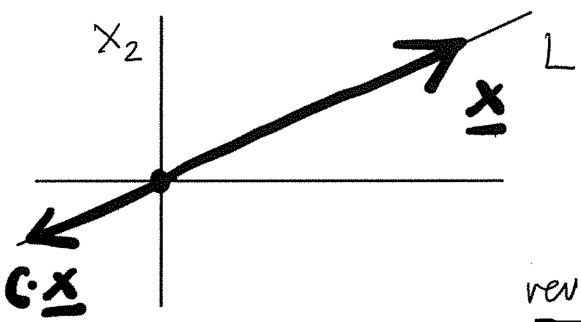


- Take two vectors $\underline{x}$ & $\underline{y}$ such that their end points are on L. (Recall that the starting point of a vector is always at the origin, by def.

- Draw $(\underline{x} + \underline{y})$. From Calculus, recall that when you add two parallel (or anti-parallel) vectors, you'll get a vector parallel to the original ones. So, as shown above, $(\underline{x} + \underline{y})$ is also on L, $\Rightarrow$ (C1) holds. ✓

Note: You didn't need to take $\underline{x}$ and $\underline{y}$ anti-parallel, as above. You could have taken them to be parallel. $(\underline{x}+\underline{y})$ would then be longer than both $\underline{x}$ & $\underline{y}$, but it will still be on L, and so the same conclusion as above would hold.

2) (C2)



- Take some $\underline{x}$ on L.
- Draw $c\cdot\underline{x}$. Again, from Calculus, $c\cdot\underline{x} \parallel \underline{x}$ (we also reviewed this in Secs. 1.7 & 3.1).

So you get $c\cdot\underline{x}$ as shown. (In this particular example, $c<0$ and $|c|<1$ — make sure you understand why; if you don't — review the very first section about vectors in your Calculus text.)

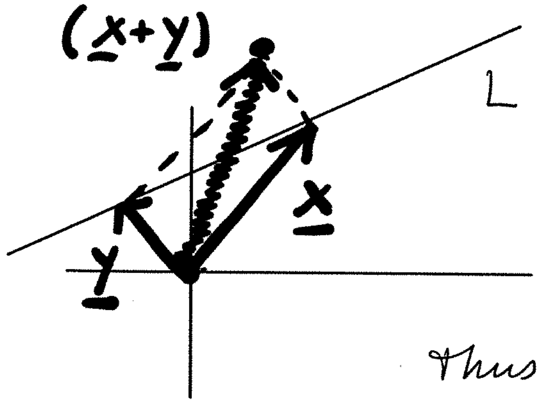
The important point is that $(c\cdot\underline{x})$ is also on L, $\Rightarrow$ (C2) holds. ✓

Note: You may notice that Method 2 (geometric) is easier to appy than Method 1 (algebraic); at least it requires less writing. Therefore,

I strongly recommend that you use the geometric method whenever possible. (Using the algebraic method is OK, too, of you are more comfortable with it.)

Ex. 4 A line **not** passing through the origin in $\mathbb{R}^2$ is **not** a vector space.

Sol'n: use the geometric method.



Check (C1):
Take $\underline{x}$ and $\underline{y}$ on L.
Then $(\underline{x}+\underline{y})$ is not on L.

Thus, (C1) FAILS, and there is no need to check (C2). L is not a vec. space.

② Subspaces of $\mathbb{R}^n$

Note: A vector space W has properties (C1) & (C2):

$$\begin{cases} (C1) & (\underline{x}, \underline{y} \text{ in } W) \Rightarrow (\underline{x}+\underline{y} \text{ in } W) \\ (C2) & (\underline{x} \text{ in } W) \Rightarrow (c \cdot \underline{x} \text{ in } W \text{ for any } c). \end{cases}$$

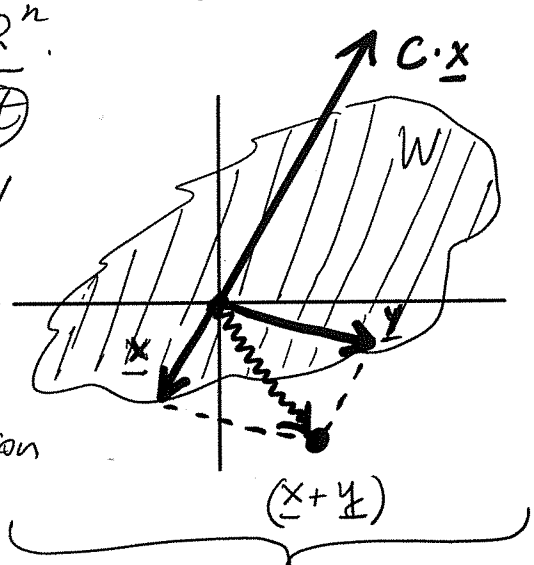
combined: $$(\underline{x}, \underline{y} \text{ in } W) \Rightarrow \Big(\underbrace{(a\underline{x} + b\underline{y})}_{\text{linear combination of } \underline{x} \,\&\, \underline{y}} \text{ in } W \text{ for any } a, b \Big)$$

In this course we'll be working with subsets W of $\mathbb{R}^n$ and will need to take linear combinations of vectors in W. We will want these linear combinations to remain in W. Thus, we'll want these subsets W to be vector spaces themselves.

Def: A subset W of $\mathbb{R}^n$ such that W itself is a vector space is called a **subspace of $\mathbb{R}^n$**.

Thus, we want to work only with those W which are subspaces of $\mathbb{R}^n$.

In other words, we do not want to work with such W where for some $\underline{x}$ & $\underline{y}$ in W, either $(\underline{x}+\underline{y})$ or $c\cdot\underline{x}$, or some other linear combination $(a\cdot\underline{x}+b\cdot\underline{y})$, may be not in W.



Example of a bad W.

> To check if a given W is a subspace, you need to check (algebraically or graphically) if (C1) & (C2) hold for W:
> (W is subspace of $\mathbb{R}^n$) $\Longleftrightarrow$ ((C1), (C2) hold for W).

- Example of a subspace of $\mathbb{R}^2$: line through origin (Ex. 3).
- Example of a non-subspace of $\mathbb{R}^2$: line not through origin (Ex. 4).

Trivial subspaces of $\mathbb{R}^2$ (or any $\mathbb{R}^n$): $\{\underline{0}\}$, $\mathbb{R}^n$.

Name: A subspace of $\mathbb{R}^n$ that is neither $\{\underline{0}\}$ nor $\mathbb{R}^n$ is called a proper subspace.

- How to quickly check if W is not a subspace:

Note that if W is a subspace, then any lin. combination $(a\underline{x}+b\underline{y})$ is in W. Take $a=b=0$. Then $0\cdot\underline{x}+0\cdot\underline{y}=\underline{\theta}$ is in W. So:

$$(W \text{ is a subspace of } \mathbb{R}^n) \Rightarrow (\underline{\theta} \text{ is in } W).$$

Take the contrapositive:

$$(\underline{\theta} \text{ is not in } W) \Rightarrow (W \text{ is not a subspace}).$$

So, the **QUICK CHECK** is:

- $(\underline{\theta}$ not in $W) \Rightarrow (W$ not a subspace of $\mathbb{R}^n)$
- $(\underline{\theta}$ in $W) \Rightarrow \ldots$ **more work needed to** determine if W is a subspace of $\mathbb{R}^n$.

3 Examples of subspaces & non-subspaces in $\mathbb{R}^2$ and $\mathbb{R}^3$.

In Ex. 3 we saw that a straight line through the origin in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.
In Ex. 4 — " — " — " — not through — " — " — " — is not a subspace of $\mathbb{R}^2$.
This can be generalized:

Claim 1: In $\mathbb{R}^2$, the only proper subspace is any straight line through the origin.

So, on **quiz or test**, you may answer the question of whether some W is a subspace of $\mathbb{R}^2$ as follows:

1. Use Sec. 3.1 to decide if W is a straight line through the origin in $\mathbb{R}^2$.
2. Use Claim 1.

Claim 2: In $\mathbb{R}^3$, the only proper subspaces are:
- any straight line through the origin;
- any plane through the origin.

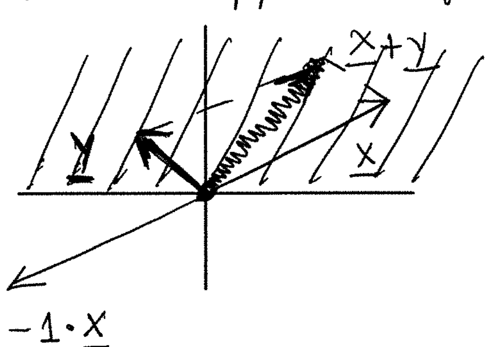
So, on a **quiz or test**, you may answer the question if some W is a subspace of $\mathbb{R}^3$ as follows:

1. Use Sec. 3.1 to decide if W is a line or plane through the origin.
2. Use Claim 2.

MUST READ Ex. 1,2 in book / 3.2 about this.

Now, let us look at examples of non-subspaces of W (the goal being to see how the closure properties (C1), (C2) can fail, not to just apply Claim 1).

Ex. 5 $\underbrace{\text{Upper-half plane}}_{\text{UHP}}$ $(x_2 \geq 0)$ is not a subspace.



Check (C1):

$\underline{x}, \underline{y}$ in UHP (point up),

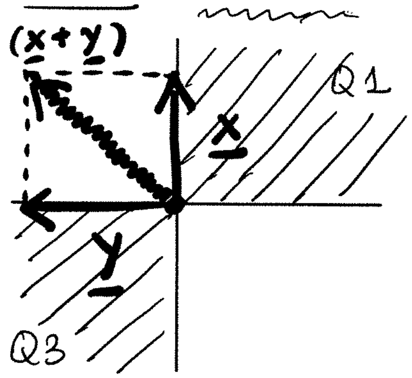
$\Rightarrow (\underline{x}+\underline{y})$ in UHP (points up). ✓

Check (C2):

$-1 \cdot \underline{x}$ not in UHP.

So, (C2) FAILS, $\Rightarrow$ UHP is not a subspace. //

Ex. 6 Union of Quadrants 1 and 3 :



$W = \{ \underline{x} : \underline{x} \text{ in } Q1 \text{ or } \underline{x} \text{ in } Q3 \}$.

Check (C1):

$\underline{x}, \underline{y}$ in W, but

$(\underline{x} + \underline{y})$ not in W.

Thus, (C1) FAILS, and there is no need to check (C2).

So, the union of Q1 & Q3 is not a subspace of $\mathbb{R}^2$.

Note: Note that in both Ex. 5 & 6, $\underline{0}$ was in W. However, this didn't make W a subspace. See again the QUICK CHECK on p. 10-8.

MUST READ: Ex. 5 in textbook (3.2).
Pay attention to word "or".
Compare it and word "and" in Ex. 2 in Notes for Sec. 3.1.

④ A simple example of why we care whether some W is or is not a vector space (or a subspace of $\mathbb{R}^n$). (OPTIONAL)

homogeneous

In Sec. 3.3 we'll show that all solutions of a homogeneous l.s. form a subspace of $\mathbb{R}^n$. (You've already seen, and will continue to see in this course and in MATH 271 that homogeneous l.s. are important.)

Here we'll illustrate, by a simple example, that solutions of only homogeneous AND linear sytems form vector spaces. In other words (recall p. 10-6), only solutions of homog. lin. systems satisfy the superposition principle: (if $\underline{x}$, $\underline{y}$ are solutions), then ($a\underline{x} + b\underline{y}$ is also a sol'n for any a, b).

Ex. 7(a) System is linear and homogeneous:

$$x - y = 0$$

Its solution: $x = y$, y = free, i.e. any vector $\underline{v} = \binom{y}{y}$.

Take some two solutions, say $\underline{v}_1 = \binom{1}{1}$, $\underline{v}_2 = \binom{2}{2}$.

Take their linear combination:

$$a\underline{v}_1 + b\underline{v}_2 = a\binom{1}{1} + b\binom{2}{2} = \binom{a+2b}{a+2b} \equiv \binom{c}{c};$$

this is also a sol'n of our l.s.

So, sol'ns of a homog. l.s. obey the linear superposition principle.

Ex. 7(b) System is linear, but non-homogeneous:

$$x - y = 1.$$

Its sol'n: $x = y+1$, y = free, or $\underline{v} = \binom{y+1}{y}$.

Take two sol'ns: $\underline{v}_1 = \binom{2}{1}$, $\underline{v}_2 = \binom{3}{2}$.

Take their linear superposition
(a simple sum): $\underline{v}_1 + \underline{v}_2 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$.
It does not fit the pattern $\begin{pmatrix} y+1 \\ y \end{pmatrix}$
and hence is not a sol'n of the l.s..

Thus, sol'ns of a linear, but non-homogeneous systems do not satisfy the lin. superposition principle (a.k.a. „are not vector spaces").

<u>Ex. 7(c)</u> System is non-linear, and homogeneous:

$$x - y^2 = 0.$$

Take two sol'ns: $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

Take their sum: $\underline{v}_1 + \underline{v}_2 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$. This doesn't fit the pattern $\begin{pmatrix} y^2 \\ y \end{pmatrix}$ and hence is not a sol'n of the given system.

$\boxed{\text{Thus, only sol'ns of } \underline{\text{linear}} \text{ \& } \underline{\text{homogeneous}} \text{ systems satisfy the linear superposition principle.}}$