Sec. 3.3 : Examples of subspaces (11-1)

① Span

<u>Def</u>: Let $S = \{\underline{v}_1, \dots \underline{v}_p\}$ be a set of vectors in $\mathbb{R}^n$. The <u>span of S</u> is the set of <u>all</u> <u>linear combinations</u> of $\underline{v}_1, \dots \underline{v}_p$:

$$Sp(S) = \{\underline{y} : \underline{y} = a_1 \underline{v}_1 + \dots + a_p \underline{v}_p, \ \underset{\text{any numbers}}{a_1, \dots a_p =} \}.$$

<u>Thm. 3</u> $Sp(S)$ is a subspace in $\mathbb{R}^n$.

<u>Note</u>: The Proof in the book establishes this claim by checking properties (c1), (c2), which, of course, is OK. However, one can see that $Sp(S)$ is a subspace of $\mathbb{R}^n$ in a much more <u>straightforward</u> way. Indeed, recall from Sec. 3.2 a subspace is a part W of $\mathbb{R}^n$ s.t.:

$$(\underline{x}, \underline{y} \text{ in } W) \Rightarrow (\text{any lin. combination } a_1\underline{x} + a_2\underline{y} \text{ is in } W).$$

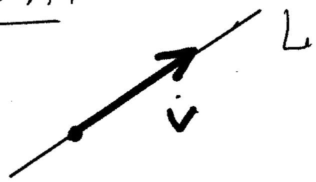
By this property, $Sp(S)$, which includes <u>all possible</u> lin. combinations of $\underline{v}_1, \dots \underline{v}_p$, must be a subspace of $\mathbb{R}^n$.

In the remainder of this topic we'll look at examples of spans in $\mathbb{R}^2$ and $\mathbb{R}^3$ and recognize them as familiar geometric objects (which we already know to be subspaces of $\mathbb{R}^{2,3}$).

Ex. 1(a) Let $\underline{v}$ be a vector in $\mathbb{R}^2$.

What is $Sp(\{\underline{v}\})$?

Sol'n:



$Sp(\{\underline{v}\}) = \{\underline{y} : \underline{y} = a \cdot \underline{v}$ for all $a\}$

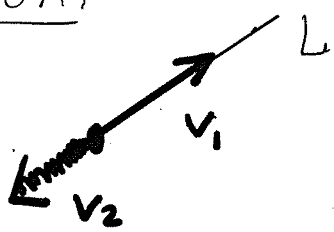
$\Rightarrow$ the line containing $\underline{v}$ (and going through $(0,0)$, because by definition, any vector starts at $(0,0)$).

So, by Claim 1 of Sec. 3.2, it's indeed a subspace of $\mathbb{R}^2$.

Ex. 1(b) $S = \{\underline{v}_1, \underline{v}_2\}$, where $\underline{v}_1 \parallel \underline{v}_2$.

What is $Sp(S)$?

Sol'n:



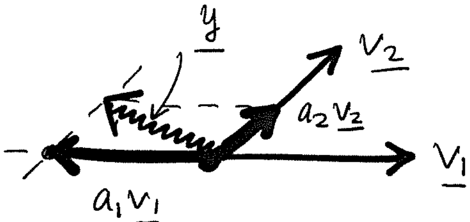
$\underline{y} = a_1 \underline{v}_1 + a_2 \underline{v}_2$

is on the same line L, containing $\underline{v}_1$ (and $\underline{v}_2$).

Ex. 1(c) $Sp(\{\underline{v}_1, \underline{v}_2\})$, $\underline{v}_1 \nparallel \underline{v}_2$.

What is $Sp(S)$?

Sol'n:



Geometrically: any $\underline{y}$ in $\mathbb{R}^2$ can be "made" as $a_1\underline{v}_1 + a_2\underline{v}_2$, as shown (see Sec. 1.7, p. 6-8).

Thus, $Sp(S) = \mathbb{R}^2$.

Algebraically: For any $\underline{y}$ in $\mathbb{R}^2$, we can solve $a_1\underline{v}_1 + a_2\underline{v}_2 = \underline{y}$, based on Thm. 13: $[\underline{v}_1, \underline{v}_2]\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \underline{y}$

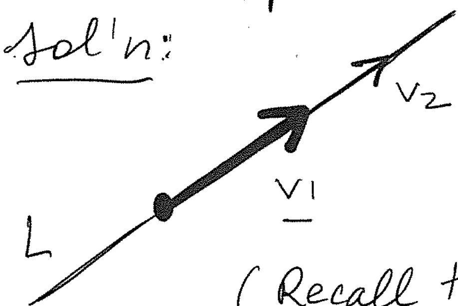
nonsingular $\longrightarrow$ $A \cdot \underline{x} = \underline{b}$

since $\underline{v}_1, \underline{v}_2$ are lin. independent.

Let us now consider $\mathbb{R}^3$.

<u>Ex. 2(a)</u> Let $S = \{\underline{v_1}; \underline{v_2}\}$, $\underline{v_1} \parallel \underline{v_2}$ in $\mathbb{R}^3$.
What is $Sp(S)$?

<u>Sol'n:</u>

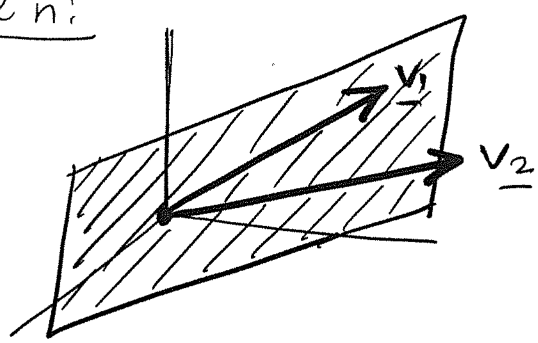


Just as in $\mathbb{R}^2$, it is the line containing both $\underline{v_1}$ & $\underline{v_2}$.

(Recall that by Claim 2 of sec. 3.2, it is a subspace of $\mathbb{R}^3$.)

<u>Ex. 2(b)</u> Let $S = \{\underline{v_1}, \underline{v_2}\}$, $\underline{v_1} \nparallel \underline{v_2}$ in $\mathbb{R}^3$.
What is $Sp(S)$?

<u>Sol'n:</u>



Similarly to Ex. 1(c), $Sp(S) =$ plane created (= made, spanned) by $\underline{v_1}, \underline{v_2}$.

Based on fact from Calculus:

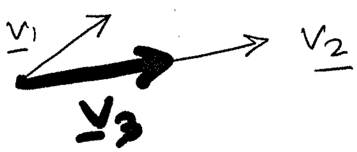
any $(a_1 \underline{v_1} + a_2 \underline{v_2})$ lies in the plane made by $\underline{v_1}, \underline{v_2}$.

<u>Ex. 3(a)</u> Let $\underline{v_1} \parallel \underline{v_2} \parallel \underline{v_3}$ in $\mathbb{R}^3$; $S = \{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$.
What is $Sp(S)$?

<u>Sol'n:</u>

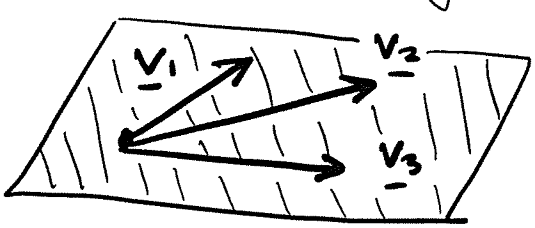
Similarly to Ex. 2(a), it is the <u>line</u> containing these three vectors.

<u>Ex. 3(b)</u> $\underline{v}_1 \nparallel \underline{v}_2 \parallel \underline{v}_3$ in $\mathbb{R}^3$; $S = \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$.
What is $Sp(S)$?
<u>Sol'n:</u>

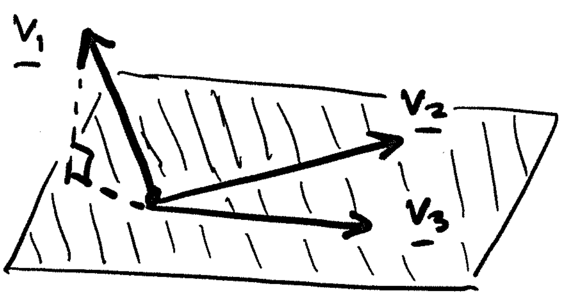


Similarly to Ex. 2(b), $Sp(S)$ = plane made by v_1 & v_2 (or v_1 & v_3).

<u>Ex. 3(c)</u> $\underline{v}_1 \nparallel \underline{v}_2 \nparallel \underline{v}_3$ in $\mathbb{R}^3$, but all three vectors <u>lie in the same plane</u>. What is $Sp(S)$?
<u>Sol'n:</u> Similarly to Ex. 2(b), $Sp(S)$ is the plane where all these thee vectors lie.



<u>Ex. 3(d)</u> $S = \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$, where $\underline{v}_1, \underline{v}_2, \underline{v}_3$ are not in the same plane. What is $Sp(S)$?



<u>Sol'n:</u>
Any $\underline{y}$ in $\mathbb{R}^3$ can be "made" as
$y = a_1 \underline{v}_1 + a_2 \underline{v}_2 + a_3 \underline{v}_3$.

<u>Geometrically</u> – See Sec. 1.7-B, p. 6-14.
<u>Algebraically</u> (similarly to Ex. 1(c)):

$$a_1\underline{v}_1 + a_2\underline{v}_2 + a_3\underline{v}_3 = \underline{y} \Rightarrow [\underline{v}_1, \underline{v}_2, \underline{v}_3]\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \underline{y}.$$

This l.s. has 1 sol'n by Thm. 13 (Chap-1), since matrix $[\underline{v}_1, \underline{v}_2, \underline{v}_3]$ is nonsingular, because its columns are linearly independent.

Point to remember (from Sec. 1.7-A):

3 vectors are lin. dependent when they are in the same plane (not when they are all parallel — that works only for 2 vectors).

So, when you are given $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ in $\mathbb{R}^3$ and asked about their span, you need to determine if they are lin. dep. or indep. (as in Sec. 1.7-A).

- $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ are **lin. dep.** $\Rightarrow Sp(S) =$ **line or plane in $\mathbb{R}^3$** (Ex. 3(a) – (c)).
- $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ are lin. indep. $\Rightarrow Sp(S) = \mathbb{R}^3$ (Ex. 3(d)).
- **If you find that $Sp(\{\underline{v}_1, \underline{v}_2, \underline{v}_3\})$ is a plane, there are two acceptable forms in which you can describe this plane:**

  1) "Plane made by two (state which ones) of the vectors" (see Ex. 3(b) or 3(c) above);

  or

  2) As done in Ex. 1 in book (Sec. 3.3).

     ↑ MUST READ.

2 Null space of a matrix

Def: Let A be $m \times n$. The null space of A is the set of all $\underline{x}$ in $\mathbb{R}^n$ s.t. $A\underline{x} = \underline{\theta}$:

$$\mathcal{N}(A) = \{\underline{x} : A\underline{x} = \underline{\theta}\}.$$

Ex. 4 Let $A = \begin{pmatrix} 1 & 2 \\ a & 3 \end{pmatrix}$. Find $\mathcal{N}(A)$.

Sol'n: From sec. 1.7-B we know that we need to consider two cases.

- $a \neq 3/2 \Rightarrow A =$ nonsingular.

  Then by the Def. of a nonsingular matrix,

  $A\underline{x} = \underline{\theta} \Rightarrow \underline{x} = \underline{\theta}, \Rightarrow \mathcal{N}(A) = \{\underline{\theta}\}$.

- $a = 3/2 \Rightarrow A =$ singular

  $A\underline{x} = \underline{\theta} \Rightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 3/2 & 3 & 0 \end{array}\right) \xrightarrow{REF} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$x_1 + 2x_2 = 0$, $x_2 =$ free $\Rightarrow x_1 = -2x_2 \Rightarrow \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2x_2 \\ 1x_2 \end{pmatrix}$.

Thus, $\mathcal{N}(A) = \left\{\underline{x} : \underline{x} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cdot x_2,\ x_2 = \text{free}\right\}$.

(Recall the vector form of sol'n of a l.s. from Sec. 1.5.)

Ex. 5 Let $A = \begin{pmatrix} 1 & 2 & 4 \\ 3/2 & 3 & 6 \end{pmatrix}$. Find $\mathcal{N}(A)$.

Sol'n: Solve $A\underline{x} = \underline{\theta}$:

$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 3/2 & 3 & 6 & 0 \end{array}\right) \xrightarrow{REF} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$

$x_1 + 2x_2 + 4x_3 = 0$.

Two forms of writing the answer:

Form 1:

$$\mathcal{N}(A) = \left\{\underline{x} : \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},\ x_1 + 2x_2 + 4x_3 = 0\right\}.$$

Geometrically, this is plane through the origin, $\perp \langle 1, 2, 4 \rangle$ (see Sec. 3.1).

Form 2: We can write the sol'n to $A\underline{x} = \underline{\theta}$ as: $x_1 = -2x_2 - 4x_3$; $x_2, x_3 = $ free.

In vector form:

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_2 - 4x_3 \\ 1 \cdot x_2 + 0 \cdot x_3 \\ 0 \cdot x_2 + 1 \cdot x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} x_3, \Rightarrow$$

$$\mathcal{N}(A) = \left\{ \underline{x} : \underbrace{\underline{x} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} x_3,}_{} \; x_2, x_3 = \text{any number} \right\}$$

Geometrically, this is the plane made by $\underline{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ and $\underline{v}_2 = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$.

Note 1: $\mathcal{N}(A) = Sp(\{\underline{v}_1, \underline{v}_2\})$.

Note 2: Both Forms describe the same plane.

Thm. 4 Let $A = m \times n$. $\mathcal{N}(A)$ is a subspace of $\mathbb{R}^n$.

Proof (OPTIONAL):

It suffices to show that if $\underline{x}$ & $\underline{y}$ are in $\mathcal{N}(A)$ then $(a \cdot \underline{x} + b \cdot \underline{y})$ is in $\mathcal{N}(A)$ for any a, b.

$\underline{x}$ in $\mathcal{N}(A) \Rightarrow A\underline{x} = \underline{\theta}$ (1)

$\underline{y}$ in $\mathcal{N}(A) \Rightarrow A\underline{y} = \underline{\theta}$ (2)

Multiply: $a \cdot (1)$, $b \cdot (2)$

$$\left.\begin{array}{l} a \cdot (1) \Rightarrow a \cdot A\underline{x} = a \cdot \underline{\theta} \Rightarrow A(a\underline{x}) = \underline{\theta} \\ b \cdot (2) \Rightarrow b \cdot A\underline{y} = b \cdot \underline{\theta} \Rightarrow A(b\underline{y}) = \underline{\theta} \end{array}\right\} \text{add:}$$

$$A(a\underline{x}) + A(b\underline{y}) = \underline{\theta} + \underline{\theta} \Rightarrow A(a\underline{x} + b\underline{y}) = \underline{\theta}$$

$\Rightarrow (a\underline{x} + b\underline{y})$ in $\mathcal{N}(A)$ by Def. of $\mathcal{N}(A)$.

3 Range of a matrix ($\mathcal{R}(A)$)

Informal Def: Range of A is the set of all vectors $\underline{y} = A\underline{x}$, where $\underline{x}$ takes on all possible values in $\mathbb{R}^n$.

(See the formal Def. of $\mathcal{R}(A)$ in book).

Note 1 Range of a matrix is similar to the range of a weapon. Indeed, range of a gun are all points where one can reach with a gun (bullet). Likewise, range of a matrix consists of all vectors that A can create from all vectors in $\mathbb{R}^n$.

Note 2 $\mathcal{R}(A) = \{\underline{y} : \underline{y} = A\underline{x} \text{ for all } \underline{x} \text{ in } \mathbb{R}^n\}$.

Equivalently, using the Key Formula:

$$A\underline{x} = [\underline{A_1}, \dots \underline{A_n}]\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{x_1 \underline{A_1} + \dots + x_n \underline{A_n}}_{\text{This is } Sp(\{\underline{A_1}, \dots \underline{A_n}\})} \quad \text{for all } x_1, \dots x_n.$$

Thus, **$\mathcal{R}(A)$ = Sp (columns of A).**

So, following Ex. 1, 2, 3 above:

- If $A = 2 \times n$, $\Rightarrow \mathcal{R}(A) =$ either line in $\mathbb{R}^2$ or $\mathbb{R}^2$
- If $A = 3 \times n$, $\Rightarrow \mathcal{R}(A) =$ either line in $\mathbb{R}^3$, or plane in $\mathbb{R}^3$, or $\mathbb{R}^3$.

$\boxed{\textbf{MUST READ on your own: Ex. 4 in book}}$

(Its method is similar to that of MUST-READ Ex. 1).

MUST NOT USE material on Row Space (pp. 183−185).