Ex. 7 Find $\mathcal{R}(A)$ for $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 7 & 6 \\ 5 & 8 & 9 \end{pmatrix}$

Sol'n: Following the method of Exs. 1 and 4 of Sec. 3.3 we ask: Are there any constraints on vector $\underline{b}$ if we solve $A\underline{x} = \underline{b}$? (Indeed, by definition, $\mathcal{R}(A)$ is the set of all possible vectors $\underline{b}$, thus we need to find out if $\underline{b}$ can be anywhere in $\mathbb{R}^3$ or must it be constrained.)

$$A\underline{x} = \underline{b} \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 4 & 7 & 6 & b_2 \\ 5 & 8 & 9 & b_3 \end{array}\right).$$

3 steps of the REF algorithm yield:

$$\left(\begin{array}{ccc|l} 1 & 2 & 3 & b_1 \\ 0 & 1 & 6 & 4b_1 - b_2 \\ 0 & 0 & 6 & 3b_1 - 2b_2 + b_3 \end{array}\right).$$

Note that this is not yet the REF of A (and not even the EF — we need to divide the last row by 6 to get the EF).

However, we can see that no matter what the components of $\underline{b}$ are, we can always determine the vector $\underline{x}$. Indeed, from the last row we have:

$0 \cdot x_1 + 0 \cdot x_2 + 6 \cdot x_3 = 3b_1 - 2b_2 + b_3,$

whence we can find x_3.

Then from the previous row we have:

$0 \cdot x_1 + 1 \cdot x_2 + 6 \cdot x_3 = 4b_1 - b_2,$

and since we know x_3, we can find x_2.

Similarly, from the 1st row, we can find x_1.

Therefore, $\underline{b} \in \mathbb{R}^3$ (no constraints), $\Rightarrow \mathcal{R}(A) = \mathbb{R}^3$.

Note 1 The same answer is arrived at if one shows that the columns of A are lin. independent. Then we know from Ex. 3(d) on p. 11-4 of these Notes that the columns of A span $\mathbb{R}^3$.

Note 2 You may swap "5" and "7" in A and see how it affects the answer.