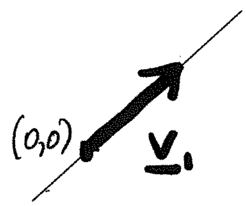
Sec. 3.4. Bases for subspaces.

① Motivation and definition

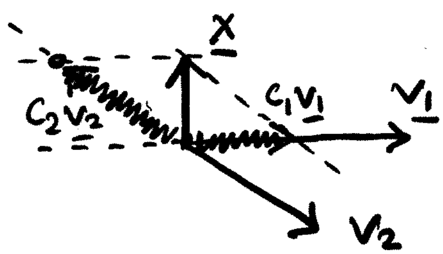
Ex. 1(a) Consider any $\underline{v}_1$ in $\mathbb{R}^2$.



$Sp(\{\underline{v}_1\})$ = line L containing $\underline{v}_1$.

$L \neq \mathbb{R}^2$, $\Rightarrow$ one vector in $\mathbb{R}^2$ does not span $\mathbb{R}^2$.

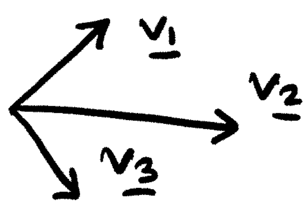
Ex. 1(b) Consider any two vectors $\underline{v}_1 \not\parallel \underline{v}_2$ in $\mathbb{R}^2$.



$Sp(\{\underline{v}_1, \underline{v}_2\}) = \mathbb{R}^2$ because any $\underline{x}$ in $\mathbb{R}^2$ can be created by a linear combination of $\underline{v}_1, \underline{v}_2$: $\underline{x} = c_1\underline{v}_1 + c_2\underline{v}_2$.

So, $\{\underline{v}_1, \underline{v}_2\}$ spans $\mathbb{R}^2$, and neither $\underline{v}_1$ or $\underline{v}_2$ is redundant (since any one of them doesn't span $\mathbb{R}^2$).

Ex. 1(c) Consider any 3 non-parallel vectors in $\mathbb{R}^2$: $\underline{v}_1, \underline{v}_2, \underline{v}_3$.



They span $\mathbb{R}^2$ (because any two of them do), but also any one of them is redundant (again, because any two of them already suffice to span $\mathbb{R}^2$). This redundancy occurs because 3 vectors in $\mathbb{R}^2$ are lin. dep. (Sec. 1.7-A):

$$\underline{v}_3 = k_1\underline{v}_1 + k_2\underline{v}_2.$$

<u>Def:</u> let W be a subspace of $\mathbb{R}^n$. A set $\{\underline{v}_1, \underline{v}_2, \dots \underline{v}_m\}$ in $\mathbb{R}^n$ is called a <u>basis for W</u> if:

- any $\underline{x}$ in W can be created as a lin. comb. of the basis vectors:

  $$\underline{x} = c_1 \underline{v}_1 + \dots c_m \underline{v}_m \quad \text{for some } c_1, \dots c_m.$$

- the set $\{\underline{v}_1, \dots \underline{v}_m\}$ is <u>lin. independent</u> (i.e., there are <u>no redundant vectors</u> in it).

(You must also read the def. of a <u>spanning set</u> on p.190.)

In <u>Ex. 1(a)</u>, $\{\underline{v}_1\}$ was <u>not</u> a basis of $\mathbb{R}^2$ because it did not span $\mathbb{R}^2$.

In <u>Ex. 1(b)</u>, $\{\underline{v}_1, \underline{v}_2\}$ <u>was</u> a basis of $\mathbb{R}^2$, because both conditions in the Definition were satisfied.

In <u>Ex. 1(c)</u>, $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ was <u>not</u> a basis of $\mathbb{R}^2$, because this set is lin. dependent.

2 <u>Coordinates in a basis</u>

If $\{\underline{v}_1, \dots \underline{v}_m\}$ is a basis of W, then any $\underline{x}$ in W can be **<u>expanded over this basis</u>**:

$$\underline{x} = c_1 \underline{v}_1 + \dots c_m \underline{v}_m .$$

The numbers $c_1, \dots c_m$ are called **coordinates of $\underline{x}$ in the basis $\{\underline{v}_1, \dots \underline{v}_m\}$.**

- Such coordinates can <u>always</u> be found because a basis <u>spans W</u>.

- The coordinates are <u>unique</u> because vectors in a basis a <u>lin. independent</u>.

(OPTIONAL: See the proof on p. 195 in book.)

<u>Note</u>: Coordinates in any <u>given</u> basis are unique. However, W has infinitely many different bases: just as in Ex. 1(b), there are ∞ many pairs of non-parallel vectors in $\mathbb{R}^2$.

<u>Q</u>: How to find coordinates of $\underline{x}$ in a given basis?

<u>Ex. 2</u> Vectors $\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\underline{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ span a plane in $\mathbb{R}^3$. Show that vector $\underline{x} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ lives in this plane.

<u>Sol'n</u>: As we know from Sec. 1.7-A,

(3 vectors are in same plane) $\Leftrightarrow$ (they are lin. <u>dep.</u>)

Therefore, we need to show that

$$\underline{x} = c_1 \underline{v}_1 + c_2 \underline{v}_2$$

By Key Formula: $[\underline{v}_1, \underline{v}_2] \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \underline{x}$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 1 & 3 & -1 \end{array}\right) \xrightarrow{\text{REF}} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} c_1 = 2 \\ c_2 = -1 \end{array}$$

coordinates of $\underline{x}$ in basis $\{\underline{v}_1, \underline{v}_2\}$

<u>In general</u>, to find coordinates of $\underline{x}$ in basis $\{\underline{v}_1, \dots \underline{v}_m\}$, follow the <u>same algorithm</u>:

$$c_1 \underline{v}_1 + \dots c_m \underline{v}_m = \underline{x} \quad \Rightarrow$$

$$[\underline{v}_1, \dots \underline{v}_m] \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} = \underline{x} \Rightarrow [\underline{v}_1, \dots \underline{v}_m \mid \underline{x}] \xrightarrow{\text{REF}} \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} = \dots$$

3) Methods of finding a basis for W

In sec. 3.3 we studied three examples of subspaces of $\mathbb{R}^n$: span, null space, and range. For each of these types of subspaces, we'll use a method suitable for this specific example.

Method 1 : Basis for a span of vectors

Ex. 3 Find a basis for $Sp(\{\underline{v}_1, \underline{v}_2, \underline{v}_3\})$ where

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix}, \quad \underline{v}_3 = \begin{pmatrix} 2 \\ 7 \\ 0 \end{pmatrix}.$$

Sol'n: Obviously, $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ span $Sp(\{\underline{v}_1, \underline{v}_2, \underline{v}_3\})$. So, it remains to determine if any of these vectors are lin. dependent on the rest of the vectors. We will then "throw away" any redundant vectors and keep only lin. indep. vectors in the basis.

1) Setup: $c_1 \underline{v}_1 + c_2 \underline{v}_2 + c_3 \underline{v}_3 = \underline{\theta}$

definition of lin. (in)dependence of vectors.

2) By Key Formula, write this as l.s. in matrix form:

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 7 \\ 3 & 7 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow[\text{matrix}]{\text{Augm.}} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 2 & 0 & 7 & 0 \\ 3 & 7 & 0 & 0 \end{array}\right) \xrightarrow{\text{REF}} \left(\begin{array}{ccc|c} 1 & 0 & 7/2 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow c_1 = -7/2\, c_3, \; c_2 = -3/2\, c_3, \; c_3 = \text{free}.$

3) Substitute into the setup eqn.:

$$\underbrace{c_1}_{-\frac{7}{2}c_3} \underline{v}_1 + \underbrace{c_2}_{-\frac{3}{2}c_3} \underline{v}_2 + c_3 \underline{v}_3 = \underline{\theta} \xrightarrow[\text{and solve for } \underline{v}_3:]{\text{Cancel by } c_3} \underline{v}_3 = \frac{7}{2}\underline{v}_1 + \frac{3}{2}\underline{v}_2$$

Therefore, $\underline{v}_3$ is lin. dep. on $\{\underline{v}_1, \underline{v}_2\}$ and hence it is not needed for a basis.

So, a basis of $Sp(\{\underline{v}_1, \underline{v}_2, \underline{v}_3\})$ is $\{\underline{v}_1, \underline{v}_2\}$.

<u>Note 1</u>: So the general algorithm is this: Find lin. <u>dependent</u> vectors in the given set (they correspond to free variables) and "throw them away". The remaining vectors for a basis for $Sp(\{\underline{v}_1, \underline{v}_2, \underline{v}_3\})$.

<u>Note 2</u>: **MUST READ Ex. 4 in Book (3.4):** It explains <u>in detail</u> why a lin. dep. vector is not needed for a basis.

<u>Note 3</u>: **MUST READ Ex. 6 in Book (3.4):** It gives an example of <u>more than one redundant</u> vector in a set.

<u>Method 2</u>: <u>Basis for a null space</u>

<u>Ex. 4</u> Find a basis for the null space of $V = [\underbrace{\underline{v}_1, \underline{v}_2, \underline{v}_3}]$ vectors from Ex. 3.

<u>Sol'n</u>: 1) Setup: definition of null space

$$V\underline{x} = \underline{\theta} \Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 2 & 0 & 7 & 0 \\ 3 & -7 & 0 & 0 \end{array}\right) \xrightarrow[\text{as in Ex. 3}]{\text{Same REF}} \begin{array}{l} x_1 = -\frac{7}{2} x_3 \\ x_2 = -\frac{3}{2} x_3 \\ x_3 = \text{free} \end{array}$$

2) $$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -7/2\, x_3 \\ -3/2\, x_3 \\ 1 \cdot x_3 \end{pmatrix} = \begin{pmatrix} -7/2 \\ -3/2 \\ 1 \end{pmatrix} \cdot x_3, \quad x_3 = \text{free}$$

3) $\mathcal{N}(V) = \{\underline{x} : \underline{x} = \begin{pmatrix} -7/2 \\ -3/2 \\ 1 \end{pmatrix} x_3,\ x_3 = \text{free}\}$,

and so the <u>basis for $\mathcal{N}(V)$</u> is $\left\{\begin{pmatrix} -7/2 \\ -3/2 \\ 1 \end{pmatrix}\right\}$.

Note 1: The calculation here is exactly the same as in Ex. 3; however, the answer is completely different. Therefore, it is essential that **you clearly explain the setup of your work!**

Note 2: **MUST READ Ex. 3 in book.**
It gives an example when the basis for a null space contains more than one vector. (See also Ex.5/Form2 in posted Notes for Sec.3.3 (p.11-7).)

Method 3: Basis for the range of matrix.

Recall from Sec. 3.3 that the range of A is the span of its columns:

$$\mathcal{R}(A) = Sp(\{\underline{A}_1, \ldots \underline{A}_n\}).$$

Using this fact, we formulate the 1st strategy.

Strategy 1: Find the basis of $Sp(\{\underline{A}_1, \ldots \underline{A}_n\})$ using method 1 above (of finding a basis for the span).

This is an EASY and convenient method!

Strategy 2: **MUST READ Ex. 5 in book.**

It is related to: Ex. 5/Notes of 3.3;
Must-read Ex. 1,4/book of 3.3.