# Sec. 3.5 Dimension

## ① Definition and examples

We know that dimensions of $\mathbb{R}^2$ and $\mathbb{R}^3$ are 2 and 3, respectively, which is just the number of components of vectors in this spaces. Similarly, the dimension of $\mathbb{R}^n$ is $n$.

But what about the dimension of a plane in $\mathbb{R}^3$? Vectors in such a plane have 3 components, but a plane is similar to $\mathbb{R}^2$, so we intuitively want to say that even in $\mathbb{R}^3$, the dimension of a plane is 2.

Now recall from Secs. 3.3 and 3.4 that a plane is spanned by any 2 non-// vectors in it. I.e., any basis of a plane contains 2 vectors.

This motivates the general definition.

Def: Let $W$ be a subspace of $\mathbb{R}^n$ and let $B = \{\underline{w}_1, \dots \underline{w}_p\}$ of $p$ vectors be a basis for $W$. The dimension of $W$ = number of vectors in a basis:

$$\dim(W) = p.$$

An obvious problem: A basis for $W$ is not unique. Then how do we know that all bases for $W$ have the same # of vectors, to make $\dim(W)$ unique?

The uniqueness of dim(W) is proved in the Corollary to Thm. 8 in the textbook (OPTIONAL): All bases for a subspace W of $\mathbb{R}^n$ contain the same number of vectors.

To find dim(W):

- recognize W as one of the 3 types of subspaces: span, null space, or range (sec. 3.3);
- find its basis by the method appropriate for this type of subspace (sec. 3.4) and count the number of vectors in this basis.

Ex. 1 Find the dimension of $Sp(\{\underline{v}_1, \underline{v}_2, \underline{v}_3\})$, where $\underline{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $\underline{v}_3 = \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix}$.

Sol'n: Since W is a span, we use Method 1 of Sec. 3.4/Notes to find its basis. (Review it.)

1) $c_1 \underline{v}_1 + c_2 \underline{v}_2 + c_3 \underline{v}_3 = \underline{\theta}$

2) $[\underline{v}_1, \underline{v}_2, \underline{v}_3] \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \underline{\theta} \Rightarrow$

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ -1 & 1 & -5 & 0 \\ 1 & 2 & -1 & 0 \end{array}\right) \xrightarrow{REF} \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} c_1 = \ldots \\ c_2 = \ldots \\ c_3 = \text{free} \end{array}$$

$\Rightarrow$ $\underline{v}_3$ is lin. dep. on $\underline{v}_1, \underline{v}_2$ and hence is not needed for a basis. So, basis of $Sp(S) = \{\underline{v}_1, \underline{v}_2\}$,
$\Rightarrow$ $\dim(Sp(S)) = 2$. 2 vectors

Ex. 2 Let $W = \left\{ \underline{x} : \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}, \begin{array}{l} x_1 + x_2 + x_3 - 2x_4 + 3x_5 = 0 \\ x_2 + 2x_3 - 6x_4 + 8x_5 = 0 \end{array} \right\}$

Find dim(W).

Sol'n: 1) Recognize W as being one of the three types of subspaces considered in Sec. 3.3. W is defined by equations of the form $A\underline{x} = \underline{\theta}$. Therefore, $W = \mathcal{N}(A)$. Therefore, we follow Method 2 described in Sec. 3.4 / Notes.

2) $A\underline{x} = \underline{\theta} \Rightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 1 & -2 & 3 & 0 \\ 0 & 1 & 2 & -6 & 8 & 0 \end{array}\right) \xrightarrow{REF}$

$\left(\begin{array}{ccccc|c} 1 & 0 & -1 & 4 & -5 & 0 \\ 0 & 1 & 2 & -6 & 8 & 0 \end{array}\right) \Rightarrow$ $\begin{array}{l} x_1 = x_3 - 4x_4 + 5x_5 \\ x_2 = -2x_3 + 6x_4 - 8x_5 \\ x_3, x_4, x_5 = \text{free}, \end{array} \Rightarrow$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_3 - 4x_4 + 5x_5 \\ -2x_3 + 6x_4 - 8x_5 \\ 1\cdot x_3 + 0\cdot x_4 + 0\cdot x_5 \\ 0\cdot x_3 + 1\cdot x_4 + 0\cdot x_5 \\ 0\cdot x_3 + 0\cdot x_4 + 1\cdot x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -4 \\ 6 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_4 + \begin{pmatrix} 5 \\ -8 \\ 0 \\ 0 \\ 1 \end{pmatrix} x_5$$

Answer:

Basis for $W = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 6 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ -8 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Note 1: **MUST READ Thm. 9**
**(properties of a p-dimensional W ← easy reading).**

Note 2: Avoid a common mistake: **Do not attempt** to solve this by manipulating the equations!!
**ALWAYS USE THE REF !!!**

## ② Nullity and rank of a matrix

Def:

Let A be $m \times n$.

Nullity of $A = \dim(\mathcal{N}(A))$.

Rank of $A = \dim(\mathcal{R}(A))$.

Note 1: $\boxed{\text{rank}(A) = \#\text{ of lin. indep. columns of } A.}$

Note 2: Ex. 4 in the book shows a method of finding rank (A) that is based on the concept of Row Space. This concept was declared in Sec. 3.3 to be outside the scope of this course. Therefore, you are **not allowed to use the method of Ex. 4/book to find rank (A) on a quiz or test.**

Instead, use the methods that we covered in Sec. 3.4.

Ex. 3 Let
$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
← Already in REF, to save time on a well-known calculation.

Find nullity (A) and rank (A).

Sol'n: 1) For nullity (A), we need a basis for $\mathcal{N}(A)$. Follow Method 2 of Sec. 3.4.

$$A\underline{x} = \underline{\theta} \Rightarrow \begin{aligned} x_1 &= -x_2 - x_4 \\ x_3 &= 0\cdot x_2 - 2x_4 - 3x_5 \end{aligned} \qquad x_2, x_4, x_5 = \text{free}$$

$$\underline{x} = \begin{pmatrix} -x_2 - x_4 + 0\cdot x_5 \\ 1\cdot x_2 + 0\cdot x_4 + 0\cdot x_5 \\ 0\cdot x_2 - 2x_4 - 3x_5 \\ 0\cdot x_2 + 1\cdot x_4 + 0\cdot x_5 \\ 0\cdot x_2 + 0\cdot x_4 + 1\cdot x_5 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} x_4 + \begin{pmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{pmatrix} x_5$$

basis for $\mathcal{N}(A)$

$$\Rightarrow \text{nullity}(A) = \dim(\mathcal{N}(A)) = \boxed{3}.$$

2) To find rank(A), need a basis for $\mathcal{R}(A)$. As explained in Sec. 3.4, for this one should use Strategy 1 of Method 3 in almost all cases, since it is much easier than Strategy 2. It is equivalent to Method 1 of finding a basis for the span of columns of A. So:

- $c_1 \underline{A_1} + c_2 \underline{A_2} + \dots + c_5 \underline{A_5} = \underline{\theta}$
- $[\underline{A_1}, \dots \underline{A_5}] \begin{pmatrix} c_1 \\ \vdots \\ c_5 \end{pmatrix} = \underline{\theta} \Rightarrow A\underline{c} = \underline{\theta}.$

Note: You end up solving the same eq. as in finding nullity(A). So, we can use its result. However, **you must always present a setup and clearly explain what you are finding.**

- From the work for $\mathcal{N}(A)$, we know that $c_2, c_4, c_5$ = free, $\Rightarrow$ $\underline{A_2}, \underline{A_4}, \underline{A_5}$ are lin. dependent on $\underline{A_1}, \underline{A_3}$. So, basis for $\mathcal{R}(A)$ is: $\underline{A_1}, \underline{A_3}$

$\Rightarrow$ rank(A) = $\boxed{2}$.

Note: nullity(A) = 3, rank(A) = 2,
\# of columns of A = 5;
3 + 2 = 5.

This is the general fact, not a coincidence:

(*) $\rightarrow$ $\boxed{\text{rank}(A) + \text{nullity}(A) = n}$ $\leftarrow$ # of columns

## Interpretation of (*):

$$\underbrace{\text{rank}(A)}_{\substack{\uparrow \\ \text{\# of lin. indep.} \\ \text{columns} \\ \text{(by Def. of rank)}}} + \underbrace{\text{nullity}(A)}_{\substack{\uparrow \\ \text{\# of free variables} \\ \text{in } A\underline{x} = \underline{\theta}}} = n \leftarrow \begin{array}{c}\text{total \# of columns} \\ = \\ \text{total \# of. variables}\end{array}$$

(see Method 2 /sec. 3.4; must-read Ex. 3 /book/3.4; Ex. 2 above)

### 1st interpretation:

$$\text{rank}(A) + \text{nullity}(A) = n$$

- rank(A): # of lin. indep. columns of A
- nullity(A): MUST BE # of lin. dep. columns of A
- n: total # of columns of A

(agrees with Method 1 of sec. 3.4)

### 2nd interpretation:

$$\text{rank}(A) + \text{nullity}(A) = n$$

- rank(A): MUST BE # of dependent (not free) variables
- nullity(A): # of independent (free) variables
- n: total # of variables in $A\underline{x} = \underline{\theta}$

(agrees with Method 2 of sec. 3.4).

**Note:**

- (Independent columns) correspond to (Dependent variables)
- (Dependent columns) correspond to (Independent variables)

This is called **duality** between columns & variables.

## ③ Two theorems about rank (A)

Thm. 10 Let A be $m \times n$.

$$\boxed{\underbrace{\operatorname{rank}(A)}_{\substack{\text{\# of lin. indep.}\\ \text{columns of } A}} = \underbrace{\operatorname{rank}(A^T)}_{\substack{\text{\# of lin. indep.}\\ \text{columns of } A^T}}}$$

$$\text{\# of lin. indep. columns of } A^T = \text{\# of lin. indep. ROWS of } A$$

So:

$$\boxed{\begin{array}{c}\text{\# of lin. indep.}\\ \underline{\text{COLUMNS}} \text{ of } A\end{array} = \begin{array}{c}\text{\# of lin. indep}\\ \underline{\text{ROWS}} \text{ of } A\end{array}}$$

Idea of proof:

0) Putting a matrix into REF doesn't change the # of independent rows or columns. So we can consider A in the REF.

1) Each independent row of A defines a dependent variable. E.g., in Ex. 3,

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \to \boxed{x_1} = -x_2 - x_4 \\ \to \boxed{x_3} = 0 \cdot x_2 - 2x_4 - 3x_5 \\ \qquad x_2, x_4, x_5 = \text{free} \end{array}$$

(dependent variables)

2) By the **duality** between columns & variables, (dependent variables) $\longleftrightarrow$ (independent columns),

$$\Rightarrow \left(\begin{array}{c}\text{\# of indep.}\\ \text{columns}\end{array}\right) = \left(\begin{array}{c}\text{\# of dep.}\\ \text{variables}\end{array}\right) = \left(\begin{array}{c}\text{\# of indep.}\\ \text{rows}\end{array}\right).$$

<u>Thm. 12</u> (obvious) Let $A$ be $n \times n$.

$$(A = \text{nonsingular}) \Longleftrightarrow (\operatorname{rank}(A) = n).$$

<u>Idea of proof:</u>

This is just a restatement of Thm. 12 of Chap. 1:

$$(A = \text{nonsingular}) \Longleftrightarrow \left( \underset{\substack{\| \\ n}}{\underline{\text{all}}} \text{ columns of } A \text{ are lin. indep.} \right)$$

$$\Longleftrightarrow \left( A \text{ has } n \text{ lin. indep. columns} \right) \Longleftrightarrow (\operatorname{rank}(A) = n).$$