Ex. 1 Gram – Schmidt orthogonalization

(The main difference between this Example and Ex. 5 & 6 in the texbook is that in this Example, I normalize each vector as soon as it has been obtained.)

- Given a basis $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$, where the vectors are given below, construct an orthonormal basis using the GS process.

$$\underline{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \quad \underline{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Sol'n: 1) $\boxed{\underline{u}_1} = \underline{v}_1 / \|\underline{v}_1\| = \dfrac{1}{\sqrt{0^2+1^2+2^2}} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \boxed{\dfrac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}}$

2) • $\underline{\tilde{u}}_2 = \underline{v}_2 - P_{\underline{u}_1}(\underline{v}_2) = \underline{v}_2 - \underline{u}_1 (\underline{u}_1^T \underline{v}_2)$

See formula from Step 2 of the GS in $\mathbb{R}^2$

- 2nd term only: $\underline{u}_1 (\underline{u}_1^T \underline{v}_2) = \dfrac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \cdot \left(\dfrac{1}{\sqrt{5}} (0, 1, 2) \cdot \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right)$

combine

$$= \frac{1}{5} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \cdot (0 + 1 + 2 \cdot 3) = \frac{7}{5} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Entire expression:

- $\underline{\tilde{u}}_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} - \dfrac{7}{5} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \overset{\text{(verify)}}{=} \begin{pmatrix} 0 \\ -2/5 \\ 1/5 \end{pmatrix} = \dfrac{1}{5} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.$

- $\underline{u}_2 = \underline{\tilde{u}}_2 / \|\underline{\tilde{u}}_2\|$. Note that the length of $\underline{\tilde{u}}_2$ does <u>not</u> matter, and so we can drop the $(1/5)$.

Then: $\boxed{\underline{u}_2} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \cdot \dfrac{1}{\sqrt{0^2 + (-2)^2 + 1^2}} = \boxed{\dfrac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}}.$

- To make sure that you didn't make an arithmetic error, it is a good idea to check if $\underline{u}_2 \perp \underline{u}_1$.

$$\underline{u}_2^T \underline{u}_1 = \frac{1}{\sqrt{5}}(0,-2,1) \cdot \frac{1}{\sqrt{5}}\begin{pmatrix}0\\1\\2\end{pmatrix} = \frac{1}{5}(0-2\cdot 1+1\cdot 2)=0 \checkmark$$

combine

3) • $\tilde{\underline{u}}_3 = \underline{v}_3 - P_{\underline{u}_1}(\underline{v}_3) - P_{\underline{u}_2}(\underline{v}_3) =$

$$= \underline{v}_3 - \underline{u}_1(\underline{u}_1^T \underline{v}_3) - \underline{u}_2(\underline{u}_2^T \underline{v}_3).$$

- 2nd term $= \frac{1}{\sqrt{5}}\begin{pmatrix}0\\1\\2\end{pmatrix} \cdot \frac{1}{\sqrt{5}}\left((0,1,2)\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right) = \frac{1}{5}\begin{pmatrix}0\\1\\2\end{pmatrix}\cdot(0+1+2) = \frac{3}{5}\begin{pmatrix}0\\1\\2\end{pmatrix}$
- 3rd term $= \frac{1}{\sqrt{5}}\begin{pmatrix}0\\-2\\1\end{pmatrix} \cdot \frac{1}{\sqrt{5}}\left((0,-2,1)\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right) = \frac{1}{5}\begin{pmatrix}0\\-2\\1\end{pmatrix}\cdot(0-2+1) = -\frac{1}{5}\begin{pmatrix}0\\-2\\1\end{pmatrix}.$
- Entire expression $= \begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{3}{5}\begin{pmatrix}0\\1\\2\end{pmatrix} - \left(-\frac{1}{5}\right)\begin{pmatrix}0\\-2\\1\end{pmatrix} \underset{\text{(verify)}}{=} \begin{pmatrix}1\\0\\0\end{pmatrix}.$

So, this is $\tilde{\underline{u}}_3$.

- Obviously, its length $=1$, and so $\boxed{\underline{u}_3 = \begin{pmatrix}1\\0\\0\end{pmatrix}.}$
- Check orthogonality: $\underline{u}_3 \perp \underline{u}_1$? $\underline{u}_3 \perp \underline{u}_2$?

$$\underline{u}_3^T \underline{u}_1 = (1,0,0)\cdot \frac{1}{\sqrt{5}}\begin{pmatrix}0\\1\\2\end{pmatrix} = 0 \checkmark$$

$$\underline{u}_3^T \underline{u}_2 = (1,0,0)\cdot \frac{1}{\sqrt{5}}\begin{pmatrix}0\\-2\\1\end{pmatrix} = 0. \checkmark$$

<u>Answer</u>: $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\} = \frac{1}{\sqrt{5}}\begin{pmatrix}0\\1\\2\end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix}0\\-2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}.$