Sec. 4.1 The Eigenvalue Problem for 2×2 matrices

① Motivation

In the first lecture of this course, we considered an example about subscribers and non-subscribers. We observed, by brute-force calculations, some interesting dynamics of their populations. Later, in Project 1, you observed a similar dynamics of populations of donating and non-donating alumni.

Here we will revisit the subscribers-non-subscribers example and explain the dynamics observed there.

Ex. 1 In a town, people either subscribe or do not subscribe to a local newspaper. Every year, the numbers of subscribers (S) and non-subscribers (N) change according to:

$$S_{n+1} = 0.7\,S_n + 0.5\,N_n$$

$$N_{n+1} = 0.3\,S_n + 0.5\,N_n\,,$$

i.e., in matrix form:

$$\begin{pmatrix} S \\ N \end{pmatrix}_{n+1} = \begin{pmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{pmatrix} \begin{pmatrix} S \\ N \end{pmatrix}_n \equiv A \cdot \begin{pmatrix} S \\ N \end{pmatrix}_n .$$

Find what happens to the numbers of subscribers and non-subscribers in, say, 20 years.

Clarification: In Project 1 you showed that

$$\begin{pmatrix} S \\ N \end{pmatrix}_n = A^n \cdot \begin{pmatrix} S \\ N \end{pmatrix}_0 .$$

Here, we want to obtain the answer **without computing A^n.**

Sol'n:

Assumption 1: Assume that there are two vectors $\underline{v_1}$ and $\underline{v_2}$ in $\mathbb{R}^2$ such that

$$\boxed{A\,\underline{v_1} = \lambda_1\,\underline{v_1}\,, \qquad A\,\underline{v_2} = \lambda_2\,\underline{v_2}}$$

(Greek letter "lambda")

This assumption does not follow from anywhere (except from subsequent calculations) and looks very strange, as it says that the effect of multiplying $\underline{v_1}$ and $\underline{v_2}$ by matrix A amount to simply multiplying them by respective scalars λ_1 and λ_2. But, you must have seen that this was indeed possible, when you did #43 in the HW for Sec. 1.6.

Assumption 2: Now assume that the above $\underline{v_1}$ and $\underline{v_2}$ form a basis in $\mathbb{R}^2$. Then any $\underline{x}$ in $\mathbb{R}^2$ can be expanded over this basis (Sec. 3.4):

$$\underline{x} = c_1 \underline{v_1} + c_2 \underline{v_2}.$$

We will take $\underline{x} = \begin{pmatrix} S \\ N \end{pmatrix}_0$ ← initial populations; then

$$\begin{pmatrix} S \\ N \end{pmatrix}_0 = c_1 \underline{v_1} + c_2 \underline{v_2}.$$

- Using Assumptions 1 & 2, we can now compute

$$\begin{pmatrix} S \\ N \end{pmatrix}_{20}.$$

We will actually start with $\begin{pmatrix} S \\ N \end{pmatrix}_1$, then $\begin{pmatrix} S \\ N \end{pmatrix}_2$, and then will notice a pattern.

$$\begin{pmatrix} S \\ N \end{pmatrix}_1 = A\begin{pmatrix} S \\ N \end{pmatrix}_0 = A(c_1 \underline{v_1} + c_2 \underline{v_2}) = c_1 A\underline{v_1} + c_2 A\underline{v_2}$$

$$\overset{\text{(Assumption 1)}}{=} c_1 \cdot \lambda_1 \underline{v_1} + c_2 \lambda_2 \underline{v_2}.$$

$$\begin{pmatrix} S \\ N \end{pmatrix}_2 = A\begin{pmatrix} S \\ N \end{pmatrix}_1 = A\left(c_1\lambda_1 \underline{v_1} + c_2\lambda_2 \underline{v_2}\right)$$

$$= (c_1\lambda_1) A\underline{v_1} + (c_2\lambda_2) A\underline{v_2} = c_1\lambda_1\lambda_1 \underline{v_1} + c_2\lambda_2\lambda_2 \underline{v_2}$$

$$= c_1\lambda_1^2 \underline{v_1} + c_2\lambda_2^2 \underline{v_2}.$$

Using this pattern, we see that

$$\boxed{\begin{pmatrix} S \\ N \end{pmatrix}_{20} = c_1 \lambda_1^{20} \underline{v_1} + c_2 \lambda_2^{20} \underline{v_2}.}$$

Thus, instead of computing A^{20}, we need to compute the simpler quantities (scalars) $\lambda_1^{20}, \lambda_2^{20}$.

What remains is:

- Justify Assumptions 1 & 2 by direct calculation (which would give us λ_1 & λ_2),

  and

- See how the above formula explains the observed fact that the populations of subscribers and non-subscribers stabilize at some constant values.

Example 1 adjourned

(2) Eigenvalues and eigenvectors of 2×2 matrices

Def ("The Eigenvalue Problem"):
Let A be $\boxed{n\times n}$. Let there exist an $\boxed{\underline{x}\neq\underline{\theta}}$ and a scalar λ (Greek letter "lambda") such that:

$$\boxed{A\,\underline{x} = \lambda\,\underline{x}},$$

then λ is called an **eigenvalue** of A, and the (nonzero!) $\underline{x}$ is called the **eigenvector** corresponding to λ.

Note: We require that $\underline{x}\neq\underline{\theta}$ because otherwise,

$$A\cdot\underline{\theta} = \underline{\theta} = (\text{any } \#)\cdot\underline{\theta}.$$

We'll now discuss how to find these λ and $\underline{x}$.

Consider $A\underline{x} = \lambda \underline{x} \Rightarrow A\underline{x} - \lambda \underline{x} = \underline{\theta}$

$\Rightarrow (A - \lambda)\underline{x} = \underline{\theta}$ .... ← NO!

Cannot subtract a scalar from a matrix!

Try again:

$A\underline{x} - \lambda \underline{x} = \underline{\theta} \Rightarrow A\underline{x} - \lambda \cdot I \underline{x} = \underline{\theta}$

$\Rightarrow \underbrace{(A - \lambda I)}_{\text{this is okay!}} \underline{x} = \underline{\theta}$

$n \times n$ identity matrix

Thus, to find eigenvalues and eigenvectors, we need to solve

$$\boxed{(A - \lambda I)\underline{x} = \underline{\theta}} \qquad (\star)$$

This involves 2 steps:

Step 1: Find all λ s.t. $(A - \lambda I)$ is singular.

Step 2: Determine the null space of $(A - \lambda I)$; i.e., for each λ, find $\underline{x}$ s.t. $(\star)$ holds.

In Ex. 1, A was 2×2, so in this Section we will carry out Steps 1 & 2 for 2×2 matrices only.

Let, in general, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$A - \lambda I = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}.$$

[$(A - \lambda I)$ = singular] $\Leftrightarrow$ [columns of $A - \lambda I$ are lin. dependent, i.e, proportional]

(for 2×2 only)

Thus, we require:

$$\frac{a-\lambda}{c} = \frac{b}{d-\lambda} \Rightarrow (a-\lambda)(d-\lambda) - bc = 0.$$

Thus,

$$\left(\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = \text{singular}\right) \Leftrightarrow \Big(\underbrace{(a-\lambda)(d-\lambda) - bc = 0}_{\text{quadratic eqn. for } \lambda.}\Big)$$

<u>Ex. 1 (resumed)</u> We had $A = \begin{pmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{pmatrix}$

<u>Step 1</u> Find when $(A-\lambda I)$ = singular.

$$\begin{pmatrix} 0.7-\lambda & 0.5 \\ 0.3 & 0.5-\lambda \end{pmatrix} = \text{singular} \Leftrightarrow (0.7-\lambda)(0.5-\lambda) - 0.3\cdot 0.5 = 0$$

$$\Rightarrow \lambda^2 - 1.2\lambda + 0.2 = 0 \Rightarrow \lambda_{1,2} = 1,\ 0.2.$$

(use the quadratic formula if inspection fails)

<u>Step 2</u> For <u>each λ above</u>, find $\underline{x}$ s.t. $(A-\lambda I)\underline{x} = \underline{\theta}$.

$\lambda = \lambda_1 = 1$

$$\begin{pmatrix} 0.7-1 & 0.5 \\ 0.3 & 0.5-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -0.3 & 0.5 & 0 \\ 0.3 & -0.5 & 0 \end{array}\right).$$ REF? No, not for 2×2.

<u>The reason:</u> The λ was <u>found</u> so as to make $(A-\lambda I)$ singular. For <u>2×2</u> matrices (and <u>only for them</u>!), this means that their columns (and hence rows! – end of Sec. 3.5) must be proportional. Since there are only 2 rows, we can then consider only one of them.

So, for <u>2×2 matrices</u>, <u>we don't need to use REF</u>.
We just need to check that the rows are indeed proportional (if not – check your arithmetic!), and then solve only the 1st equation:

$$-0.3x_1 + 0.5x_2 = 0 \Rightarrow x_1 = (5/3)x_2, \ x_2 = \text{free}.$$

Thus, if we denote the 1st eigenvector as $\underline{v_1}$:

$$(A - \lambda_1 I)\underline{v_1} = \underline{\theta} \Rightarrow \boxed{\underline{v_1} = \begin{pmatrix} 5/3 \\ 1 \end{pmatrix} x_2, \ x_2 = \text{free}}$$

$\lambda = \lambda_2 = 0.2$

$$\begin{pmatrix} 0.7-0.2 & 0.5 \\ 0.3 & 0.5-0.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.3 \end{pmatrix}$$

rows are proportional ✓

$$0.5x_1 + 0.5x_2 = 0 \Rightarrow x_1 = -x_2, \ x_2 = \text{free}$$

$$(A - \lambda_2 I)\underline{v_2} = \underline{\theta} \Rightarrow \boxed{\underline{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} x_2, \ x_2 = \text{free}}$$

Note that "x_2 for $\underline{v_1}$" and "x_2 for $\underline{v_2}$" are <u>not</u> related to each other, so a more correct way to write the answer:

$$\boxed{\begin{array}{lll} A\underline{v_1} = \lambda_1 \underline{v_1} & \Rightarrow & \underline{v_1} = \begin{pmatrix} 5/3 \\ 1 \end{pmatrix} \cdot a, \ a = \text{free} \\ A\underline{v_2} = \lambda_2 \underline{v_2} & \Rightarrow & \underline{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot b, \ b = \text{free} \end{array}}$$

<u>Note</u>: **An eigenvector is <u>always</u> found up to a scalar factor c.**
Indeed, if $\underline{v}$ is an eigenvector, then so is $(c \cdot \underline{v})$ for any $c \neq 0$:

$$(A\underline{v} = \lambda \underline{v}) \Leftrightarrow (cA\underline{v} = c\lambda \underline{v}) \Leftrightarrow (A(c\underline{v}) = \lambda(c\underline{v})) \Rightarrow (c\underline{v}) \text{ is an eigenvector.}$$

We can now finish Ex. 1.

As in Assumptions 1 & 2 on pp. 17-1, 2, we have found $\underline{v_1}$ & $\underline{v_2}$ such that:

A1: $A\underline{v_1} = \lambda_1 \underline{v_1}$, $A\underline{v_2} = \lambda_2 \underline{v_2}$, and

A2: $\underline{v_1}$ & $\underline{v_2}$ form a basis in $\mathbb{R}^2$ ($\underline{v_1} \not\parallel \underline{v_2}$).

Then:
$$\begin{pmatrix} S \\ N \end{pmatrix}_0 = c_1 \underline{v_1} + c_2 \underline{v_2} \quad \left(\text{E.g., } \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \frac{3}{8}\begin{pmatrix} 5/3 \\ 1 \end{pmatrix} + \frac{1}{8}\begin{pmatrix} -1 \\ 1 \end{pmatrix}. \right)$$

Then (p. 17-3):
$$\begin{pmatrix} S \\ N \end{pmatrix}_{20} = c_1 \lambda_1^{20} \begin{pmatrix} 5/3 \\ 1 \end{pmatrix} + c_2 \lambda_2^{20} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = c_1 \cdot 1^{20} \cdot \begin{pmatrix} 5/3 \\ 1 \end{pmatrix} + c_2 \cdot 0.2^{20} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$= c_1 \cdot 1 \cdot \begin{pmatrix} 5/3 \\ 1 \end{pmatrix} + c_2 \cdot (\approx 0) \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \approx c_1 \cdot \begin{pmatrix} 5/3 \\ 1 \end{pmatrix}.$$

This shows: $S_{20}/N_{20} \approx 5/3$, as found in Ex. 1 of Sec. 1.1!

end of Ex. 1

The issues that we have exposed but did not yet address/resolve are:

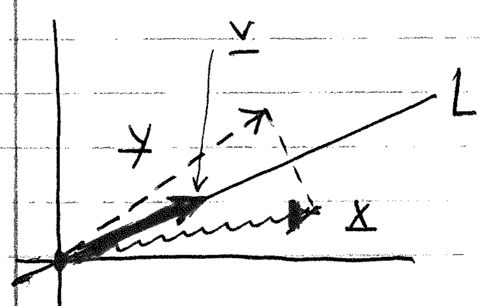
(1) Can we be sure that for another A, the eigenvectors $\underline{v_1}$ & $\underline{v_2}$ will form a basis?

(2) How do we generalize this process (= finding of the eigenvalues & eigenvectors) for a $n \times n$ matrix as opposed to just 2×2? E.g. for $n=3$?

We'll begin addressing this in Sec. 4.2.

3 Geometric meaning of eigenvectors

Ex. 2(a) Show that the matrix of reflection about a line in $\mathbb{R}^2$ always has an eigenvalue $\lambda = 1$.



Sol'n: Let A be the matrix of our l.t. T.

1) For any vector $\underline{x}$: $T(\underline{x}) = \underline{y}$ ← some $\underline{y}$ (see picture)

$\Rightarrow \boxed{A\underline{x} = \underline{y}}$.

2) For the eigenvector $\underline{v}$: $A\underline{v} = \lambda \underline{v}$ ← for some λ

$\boxed{A\underline{v} = 1 \cdot \underline{v}}$ ← for $\lambda = 1$

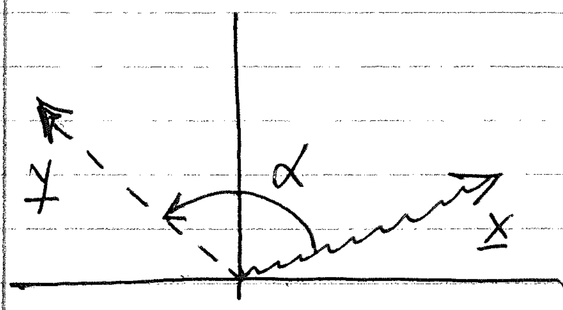
3) Combine 1) & 2): $\boxed{T(\underline{v})} \underset{\text{for any } \underline{v}}{=} A\underline{v} \underset{\text{for this } \underline{v}}{=} \boxed{1 \cdot \underline{v}}$, i.e.

$\boxed{T(\underline{v}) = \underline{v}}$ ← Thus, if $\underline{v}$ is an eigenvector of the the reflection matrix A with $\lambda = 1$, then the reflection does not change $\underline{v}$!

4) Looking at the figure, we see that such a vector indeed exists: it is any vector along the reflection line L.

Thus, we have found a vector $\underline{v}$ that satisfies $T(\underline{v}) = \underline{v}$. Then, tracing back the steps: $T(\underline{v}) = \boxed{A\underline{v} = \underline{v}}$, we see that this $\underline{v}$ is an eigenvector whose eigenvalue $\lambda = 1$.

Ex. 2(b) Show that the matrix of a nonzero rotation (by angle $\alpha \neq 0, 360°$, etc.) does not have an eigenvalue $\lambda = 1$.



Sol'n: We closely follow the steps of Ex. 2(a). Let A be the matrix of rotation.

1) For any vector $\underline{x}$: $T(\underline{x}) = \underline{y}$ ← some $\underline{y}$; see picture

$\Rightarrow A\underline{x} = \underline{y}$.

2) For an eigenvector $\underline{v}$: $A\underline{v} = \lambda \cdot \underline{v}$ ← for some λ

Let's assume that we can find this $\underline{v}$, and then prove ourselves wrong. → $[A \cdot \underline{v} = 1 \cdot \underline{v}$ ← for $\lambda = 1$

3) Combining 1) & 2): $\boxed{T(\underline{v}) = A\underline{v} = 1 \cdot \underline{v}}$

for any $\underline{v}$ for the $\underline{v}$ in 2)

$\Rightarrow \boxed{T(\underline{v}) = \underline{v}}$ ← Thus, if $\underline{v}$ is the eigenvector of the rotation matrix A (with $\lambda = 1$), then the rotation does not change it!

4) Looking at the figure, we see that such a vector cannot exist: rotation changes any vector!

Thus, our assumption that a $\underline{v}$ with $A\underline{v} = 1 \cdot \underline{v}$ exists, was wrong, $\Rightarrow \lambda = 1$ cannot be an eigenvalue!