Sec. 4.4: Eigenvalues and the characteristic polynomial

① The characteristic polynomial

Recall from Sec. 4.1 **the eigenvalue problem**:
For a given A, find $\underline{x} \neq \underline{\theta}$ s.t. $A\underline{x} = \lambda \underline{x}$;
then $\underline{x}$ is an eigenvector of A and λ is its eigenvalue.
As we discussed in Sec. 4.1, solving this requires 2 steps:

<u>Step 1</u>: Find all scalars λ s.t. $(A-\lambda I) =$ singular.

<u>Step 2</u>: For each of these λ, find $\underline{x} \neq \underline{\theta}$ s.t.
$(A-\lambda I)\underline{x} = \underline{\theta}$.

In this Section we focus on Step 1.
Compared to Sec. 4.1, <u>now</u> we can determine if $(A-\lambda I)$ is singular for $A = n\times n$, not just 2×2.

<u>Ex. 1</u> Find the eigenvalues of

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 4 & -2 & 4 \\ 1 & 0 & 3 \end{pmatrix}$$

<u>Sol'n</u>: Solve $\det(A-\lambda I) = 0$.

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 4 & -2-\lambda & 4 \\ 1 & 0 & 3-\lambda \end{vmatrix} \underset{\uparrow}{=} 0 + (-1)^{2+2} \cdot (-2-\lambda) \cdot \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} + 0$$

cofactor expand w.r.t. column 2 since it has the most zeros

$$= 1 \cdot (-2-\lambda) \cdot \left((1-\lambda)(3-\lambda) - (-1) \right) = 0$$

Note 1: Once you got some factor ($(-2-\lambda)$ in this case) multiplying everything else,

DO NOT DISTRIBUTE THAT FACTOR! This is because you want to find zeros of your $\det(A-\lambda I)$, and that factor already gives you one zero:

$$-2-\lambda=0 \Rightarrow \lambda_1=-2.$$

Continuing with Ex. 1:

$$\det(A-\lambda I) = (-2-\lambda)\cdot(\lambda^2-4\lambda+4) = 0$$

We have already found $\lambda_1=-2$. Two more zeros come from $(\lambda^2-4\lambda+4)=0 \Rightarrow (\lambda-2)^2=0$.

So, it turns out that we have a double (repeated) zero: $\lambda_2=\lambda_3=2$.

So, the answer: $\lambda_1=-2$, $\lambda_2=\lambda_3=2$.

Note 2: What would happen if you had not followed the advice to expand over the row or column with most zeros and had just expanded w.r.t. the 1st row? You'd get: $\det(A-\lambda I)=$

$$= (-1)^{1+1}\cdot(1-\lambda)\begin{vmatrix} -2-\lambda & 4 \\ 0 & 3-\lambda \end{vmatrix} + 0 + (-1)^{1+3}\cdot(-1)\cdot\begin{vmatrix} 4 & (-2-\lambda) \\ 1 & 0 \end{vmatrix} =$$

$$= (1-\lambda)\left((-2-\lambda)(3-\lambda)-0\right) + (-1)\left(0-(-2-\lambda)\right) = 0$$

In the above expression, you got **3 bad things:**

1) You no longer have "one factor multiplying everything else", so you cannot find λ_1 by inspection.
2) You have to expand all the terms (and this is very error-prone!).
3) When you expand and collect all the like powers of λ, you will get a cubic polynomial: $-\lambda^3+2\lambda^2+4\lambda-8=0$, and will need to find its roots (which is often quite hard).

So, to avoid these bad things happen to you, think before you compute your determinant!

Observations from Ex. 1:

- Our A is 3×3.
- $\det(A-\lambda I)=(-2-\lambda)(\lambda-2)^2$ is a polynomial of degree 3.
- It has 3 zeros (two of which happened to coincide).

These observations reflect the general fact:

Thms. 9 & 10 Let A be $n\times n$. Then:

- $\det(A-\lambda I)=p_n(\lambda)$ ← polynomial of degree n;
- It is called **the characteristic polynomial of A.**
- It has exactly n zeros (a.k.a. "roots"), some of which may coincide (= be repeated).
- These roots are the eigenvalues of A.

Note 1 The number of times that a root $\lambda=\lambda_K$ is repeated is called the **algebraic multiplicity** of the eigenvalue λ_K.

E.g., in Ex. 1:

alg. multiplicity of $\lambda=-2$ is 1;

alg. multiplicity of $\lambda=2$ is 2.

Note 2 Some of the roots may be complex:

Ex. 2 Find eigenvalues of $A=\begin{pmatrix} 1 & 2 & 3 \\ -2 & 1 & 4 \\ 0 & 0 & 5 \end{pmatrix}$.

Sol'n: Expand $\det(A-\lambda I)$ w.r.t. row 3:

$$\det(A-\lambda I)=\begin{vmatrix} 1-\lambda & 2 & 3 \\ -2 & 1-\lambda & 4 \\ 0 & 0 & 5-\lambda \end{vmatrix}=(5-\lambda)\left((1-\lambda)^2-(-4)\right)=0$$

$\Rightarrow \lambda_1=5$, and $(1-\lambda)^2+4=0 \Rightarrow (\lambda-1)^2=-4$

$\Rightarrow \lambda-1=\pm 2i \Rightarrow \lambda=1\pm 2i$.

(If the matrix A has only real entries, the complex eigenvalues can occur only in complex conjugate pairs: $\lambda=a\pm i\cdot b$.)

In this course we will not deal with complex eigenvalues. However, in the Bonus of Project 5 you will be given the opportunity to explore their meaning in linear transformations.

(2) General properties of eigenvalues

Thm. 11 Let $A = n\times n$ and λ be an eigenvalue of A.
Then: (i) λ^k is an eigenvalue of A^k for any $k \geq 2$;
(ii) If $A =$ nonsingular, then $(1/\lambda) \equiv \lambda^{-1}$ is an eigenvalue of A^{-1}.
(iii) If r is any scalar, then $(\lambda + r)$ is an eigenvalue of $(A + rI)$.

Note 1 Combining (i) & (ii), one can show that if A=nonsingular, λ^{-k} is an eigenvalue of $(A^{-1})^k$ for $k \geq 2$.

Proof of (i) (Compare with an old HW problem: #43 of sec. 1.6.)

We'll do the proof for $k=2$; for $k \geq 3$ it follows the same pattern.

Given: $A\underline{x} = \lambda \underline{x}$ for some $\underline{x} \neq \underline{\theta}$

Want: $A^2 \underline{y} = \lambda^2 \underline{y}$ for some $\underline{y} \neq \underline{\theta}$ (where $\underline{y}$ may or may not $= \underline{x}$)

$$A\underline{x} = \lambda\underline{x} \Rightarrow A(A\underline{x} = \lambda\underline{x}) \Rightarrow (AA)\underline{x} = A\lambda\underline{x}$$

$$\Rightarrow A^2\underline{x} = \lambda(A\underline{x}) \Rightarrow A^2\underline{x} = \lambda\cdot\lambda\underline{x} \Rightarrow A^2\underline{x} = \lambda^2\underline{x}$$

Note 2: A naive method: $(A\underline{x})^2 = (\lambda\underline{x})^2$ is **wrong**, because both $A\underline{x}$ and $\lambda\underline{x}$ are vectors, and one **cannot square a vector** (dimensions won't work!).

Note 3: By the same token, one cannot prove (ii) by: $A\underline{x} = \lambda \underline{x} \Rightarrow \frac{1}{A\underline{x}} = \frac{1}{\lambda \underline{x}} \Rightarrow A^{-1}\underline{x} = \lambda^{-1}\underline{x}$. One cannot divide by a vector...

Proof of (ii)

Given: 1) $A\underline{x} = \lambda \underline{x}$ for some $\underline{x} \neq \underline{\theta}$

2) A = nonsingular ← Need this for A^{-1} to exist.

Want $A^{-1}\underline{y} = \lambda^{-1}\underline{y}$ for some $\underline{y} \neq \underline{\theta}$.

$$A\underline{x} = \lambda \underline{x} \Rightarrow A^{-1}(A\underline{x} = \lambda \underline{x}) \Rightarrow (A^{-1}A)\underline{x} = A^{-1}\lambda \underline{x}$$

$$\Rightarrow I\underline{x} = \lambda A^{-1}\underline{x} \Rightarrow \frac{1}{\lambda}\underline{x} = A^{-1}\underline{x}. \quad //$$

Proof of (iii) — at home (see also Ex. 3 below).

Ex. 3 Let a 3×3 matrix A have the eigenvalues $\lambda = 1, -1, -2$. Find the eigenvalues of $B = A^2 + A - 2I$.

Sol'n: Let $\underline{x}$ be the eigenvector of A corresponding to an eigenvalue λ:

$$A\underline{x} = \lambda \underline{x}$$

Consider $B\underline{x} = (A^2 + A - 2I)\underline{x} =$

$$= A^2\underline{x} + A\underline{x} - 2I\underline{x} \overset{\text{(i) of}}{\underset{\text{Thm. 11}}{=}} \lambda^2\underline{x} + \lambda\underline{x} - 2\underline{x}$$

$$= (\lambda^2 + \lambda - 2)\underline{x}.$$

Therefore:

- $\underline{x}$ is also an eigenvector of B, with the eigenvalue $(\lambda^2+\lambda-2)$.
- If λ_A is an eigenvalue of A, then $(\lambda_A^2+\lambda_A-2) \equiv \lambda_B$ is an eigenvalue of B.

So:

$\underline{\lambda_A = 1} \Rightarrow \lambda_B = 1^2+1-2 = 0$

$\underline{\lambda_A = -1} \Rightarrow \lambda_B = (-1)^2+(-1)-2 = -2$

$\underline{\lambda_A = -2} \Rightarrow \lambda_B = (-2)^2+(-2)-2 = 0$ //

③ More Theorems about eigenvalues

<u>Thm. 12</u>. Let A be $n\times n$. Then A and A^T have the same eigenvalues.

<u>Proof</u>: According to Thms 9 & 10, it will suffice to show that A & A^T have the same characteristic polynomial.

$$p_{A^T}(\lambda) \equiv \det(A^T-\lambda I) \overset{I^T=I}{=} \det(A^T-\lambda I^T) =$$
$$= \det((A-\lambda I)^T) \underset{\substack{\uparrow \\ \text{property of det's} \\ \text{sec. 4.2}}}{=} \det(A-\lambda I) = p_A(\lambda).$$

q.e.d.

<u>Note</u>: Even though the eigen<u>values</u> of A & A^T are the same, their eigen<u>vectors</u> are, in general, different.

<u>Thm. 13</u> Let A be $n \times n$.

$$\boxed{(A = \text{singular}) \iff (A \text{ has a zero eigenvalue})}$$

<u>Proof</u> – Method 1:

$(A = \text{singular}) \underset{\text{Thm. 3}}{\iff} (\det A = 0) \iff$

$(\det(A - 0 \cdot I) = 0) \underset{\text{Thms. 9 \& 10}}{\iff} (\lambda = 0 \text{ is an eigenvalue of } A)$. //

<u>Proof</u> – Method 2:

$(A = \text{singular}) \underset{\text{definition, Sec. 1.7}}{\iff} (A\underline{x} = \underline{\theta} \text{ for some } \underline{x} \neq \underline{\theta}) \iff$

$(A\underline{x} = 0 \cdot \underline{x} \text{ for some } \underline{x} \neq \underline{\theta}) \underset{\text{definition, Sec. 4.1}}{\iff}$

$(\lambda = 0 \text{ is an eigenvalue of } A)$. //

<u>Thm. 14</u> Let T be $n \times n$ <u>triangular</u> (see below) matrix. Then it eigenvalues equal its diagonal entries.

<u>Proof</u> = Ex. 4 (for 3×3, but same for $n \times n$).

Find the eigenvalues of $T = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 7 \end{pmatrix}$.

This matrix is called <u>triangular</u> (more precisely – upper triangular) because all its entries below **the upper triangle** are zero.

$\det(T - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & 3-\lambda & 4 \\ 0 & 0 & 7-\lambda \end{vmatrix} =$ cofactor expand w.r.t. column 1

$= (1-\lambda) \cdot \begin{vmatrix} 3-\lambda & 4 \\ 0 & 7-\lambda \end{vmatrix} + 0 + 0 = (1-\lambda)\big((3-\lambda)(7-\lambda) - 0\big)$

$= (1-\lambda)(3-\lambda)(7-\lambda) = 0$.

By Thm. 9 & 10, these are the eigenvalues of T. //