Sec. 4.5: Eigenvectors and eigenspaces

The following Example does not introduce any new concepts, but sets the stage for the new topics that will be considered in this Section.

Ex. 1 Find eigenvectors of $A=\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}$.

Sol'n: 1) Find eigenvalues first.

$$\det(A-\lambda I)=\begin{vmatrix}1-\lambda & 2\\ 2 & 4-\lambda\end{vmatrix}=0 \Rightarrow$$

$$(1-\lambda)(4-\lambda)-4=0 \Rightarrow \lambda(\lambda-5)=0 \Rightarrow$$

$\lambda_1=0$, $\lambda_2=5$.

Note 1 Since A has an eigenvalue $=0$, then by Thm. 13 it is singular. This is consistent with this A having proportional rows & columns.

2) Find eigenvectors for each λ.

$\lambda_1=0$ Solve $\begin{pmatrix}1-0 & 2\\ 2 & 4-0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$

Note 2 This matrix is singular by design. This means that one needs to solve not 2, but $2-1=1$ equation. Thus, when finding an eigenvector of a 2×2 matrix, one can skip the REF and solve only 1 eq.

Here, $1 \cdot a + 2 \cdot b = 0 \Rightarrow a = -2b$, $b =$ free.

$\underline{v}_1 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} b$, $b =$ free.

Note 3 As we discussed in sec. 4.1, the constant b can be choosen arbitrariby ($\neq 0$!), based only on convenience. Indeed, if $\underline{v}$ is an eigenvector of A, then so is $a \cdot \underline{v}$ for any scalar $a \neq 0$:

$$A\underline{v} = \lambda \underline{v} \Rightarrow a(A\underline{v} = \lambda \underline{v}) \Rightarrow aA\underline{v} = a\lambda \underline{v}$$

$\Rightarrow A(a\underline{v}) = \lambda(a\underline{v})$, which by definition means that $(a\underline{v})$ is an eigenvector of A.

Thus, taking $b = 1$: $\lambda_1 = 0$, $\underline{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

$\lambda_2 = 5$ $\quad \begin{pmatrix} 1-5 & 2 \\ 2 & 4-5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Following Note 2 we consider only the 1st equation: $-4a + 2b = 0 \Rightarrow a = b/2$, $b =$ free.

Thus, $\underline{v}_2 = \begin{pmatrix} b/2 \\ b \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} b$, and now it is convenient to take $b = 2$ (see Note 3 above).

Thus: $\lambda_2 = 5$, $\underline{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Note 4: Recall that an **eigenvector** $\underline{v} \neq \underline{\theta}$ by definition. If in your calculations you find $\underline{v} = \underline{\theta}$, you must have made an error (either in λ or in REF).

Try to go back and fix it!

Observation: This A has two distinct eigenvalues, and its eigenvectors are lin. independent.

Q: Is this so for any $n \times n$ matrix?

Thm. 15 Let $\underline{v}_1, \dots \underline{v}_k$ be eigenvectors of an $(n \times n)$ matrix A corresponding to **distinct** eigenvalues $\lambda_1, \dots \lambda_k$:

$$A\underline{v}_j = \lambda_j \underline{v}_j, \quad j = 1, 2, \dots k \leq n,$$

and $\lambda_j \neq \lambda_i$ for $j \neq i$.

Then $\{\underline{v}_1, \dots \underline{v}_k\}$ are **lin. independent**.

Proof: see book.

Corollary: Let A be $(n \times n)$. If A has n distinct eigenvalues, then its eigenvectors form a lin indep. set (and hence a basis) in $\mathbb{R}^n$.

Q: What happens in a situation where some of the eigenvalues are repeated?

A: Then to one eigenvalue there may (but not necessarily does!) correspond more than one lin. indep. eigenvector.

The Examples below will illustrate both of these possibilities. But before, we will need to introduce a new notation.

First, note:

- All eigenvectors corresponding to an eigenvalue λ form the null space of $(A - \lambda I)$.

Def: Let $A = (n \times n)$; let λ be an eigenvalue of A. Then $\mathcal{N}(A - \lambda I)$ is called the **eigenspace of λ** and is denoted by E_λ. Furthermore,

$\dim(E_\lambda) \equiv$ geometric multiplicity of λ.

Any e-value has at least 1 e-vector

$\boxed{\text{So, geometric multiplicity} \geq 1 \text{ always.}}$

Ex. 2 Determine algebraic and geometric multiplicities of the eigenvalues of:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Sol'n: A is triangular, $\Rightarrow$ by Thm. 14, eigenvalues of A are found on the main diagonal: $\lambda_1 = \lambda_2 = \lambda_3 = 1$. So:

algebraic multiplicity of $(\lambda = 1)$ is 3.

Now, find the eigenvectors:

$$(A - 1 \cdot I)\underline{x} = \underline{\theta} \Rightarrow \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_2 = 0 \\ x_1, x_3 = \text{free} \end{array}$$

$$\Rightarrow \underline{x} = \begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

basis for $E_{\lambda=1}$,

$\Rightarrow$ **geom. multiplicity of λ is $\dim(E_{\lambda=1}) = 2$.**

Note: (geom. multiplicity) cannot be $>$ (algebr. multiplicity) for any eigenvalue.
However, the opposite is possible:
(geom. multiplicity) can be $<$ (alg. multiplicity).

So: $\boxed{1 \le \text{geom. multiplicity} \le \text{alg. multiplicity}}$.

Def: When (geom. multiplicity) $<$ (alg. multiplicity) for at least one eigenvalue λ of matrix A, this A is called **defective**.

Q: Why such a "bad" name?

A: Because then an $n \times n$ matrix A has fewer than n lin. indep. eigenvectors $\Rightarrow$ these eigenvectors cannot form a basis in $\mathbb{R}^n$.
Then, we cannot use the method of Sec. 4.1, i.e. cannot write, for an arbitrary $\underline{x}$ in $\mathbb{R}^n$, that $A\underline{x} = A(c_1\underline{v_1} + \dots + c_n\underline{v_n}) = c_1\lambda_1\underline{v_1} + \dots + c_n\lambda_n\underline{v_n}$.

The matrix A from Ex. 2 is just such a matrix. Its two eigenvectors do not form a basis in $\mathbb{R}^3$.

Finally, let's show that if a matrix has repeated eigenvalues, it does not always imply that it is defective.

Ex. 3 Find all eigenvectors of $A = I$.

Sol'n: This is a diagonal (and hence triangular) matrix, and all its eigenvalues are on the diagonal: $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

To find eigenvectors, solve

$$(A - 1 \cdot I)\,\underline{v} = \underline{\theta} \Rightarrow (I - I)\,\underline{v} = \underline{\theta}$$

$$\Rightarrow \mathcal{O}\,\underline{v} = \underline{\theta}, \text{ or } \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This imposes no restrictions on x_1, x_2, x_3, $\Rightarrow$

$$\underline{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

3 eigenvectors of A,

$\Rightarrow \dim(E_{\lambda=1}) = 3$.

Thus, in this case,

(geom. multiplicity) = (algebr. multiplicity).