Sec. 4.7 Similarity and diagonalization. Part I

① Similarity

Def. The $n \times n$ matrices A and B are similar (or related by a similarity transformation) if there is a nonsingular S s.t.

$$B = S^{-1} A S. \qquad (1)$$

Note: Rewrite (1) as:

$$SB = \underbrace{SS^{-1}}_{I} AS \Rightarrow$$

$$SB = IAS \Rightarrow \quad SB = AS \qquad (2)$$

$$\Rightarrow \quad SBS^{-1} = A\underbrace{SS^{-1}}_{I} \Rightarrow$$

$$SBS^{-1} = A. \qquad (3)$$

Thus, all of (1), (2), (3) are equivalent.

Q: How are the eigenvalues of similar matrices related?

$p_A(\lambda) = \det(A - I\lambda)$.

$p_B(\lambda) = \det(B - I\lambda) = \det(S^{-1}AS - \lambda I)$

$= \det(S^{-1}AS - \lambda S^{-1} I S)$

$= \det(S^{-1}(A - \lambda I)S) = \det(S^{-1})\det(A-\lambda I)\det(S)$

#25, Sec. 4.2 $\Rightarrow$ $\frac{1}{\cancel{\det(S)}} \det(A - \lambda I) \cancel{\det(S)} = \det(A - \lambda I)$

$= p_A(\lambda)$.

Consequently, we have:

Theorem 18:

If A and B are similar matrices, then their eigenvalues are the same. Moreover, these eigenvalues have the same algebraic multiplicities.

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Note 1: The eigenvector of similar matrices are also related. E.g., let $B = S^{-1}AS$ and $\underline{x}$ be an eigenvector of A.

Then:

$$A\underline{x} = \lambda \underline{x} \overset{(\text{by } (3))}{\Longrightarrow}$$

$$SBS^{-1}\underline{x} = \lambda \underline{x} \Rightarrow$$

$$\underbrace{\cancel{S^{-1}S}}_{I} BS^{-1}\underline{x} = S^{-1}\lambda \underline{x} \Rightarrow$$

$$B(S^{-1}\underline{x}) = \lambda(S^{-1}\underline{x}).$$

Thus, $(S^{-1}\underline{x})$ is an eigenvector of B.

Note 2:

Thus, the eigenvalues of similar matrices have not only algebraic, but also geometric multiplicities equal.

Note 3: Thm. 18 says:

"A similar to B" $\Rightarrow$ "eigenvalues of A & B are equal".

The converse statement (i.e., "$\Leftarrow$") is not true in general:

Ex. 1

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

$p_A(\lambda) = (1-\lambda)^2 = p_B(\lambda)$, $\Rightarrow$ $\lambda_A = \lambda_B = 1$ (double eigenvalue)

But there is no S s.t. $B = S^{-1}AS$. Indeed, since $A = I$, this would have meant $B = S^{-1}IS = I = A$, which is not true.

② Diagonalization

Motivation.

1. We just showed that:
   (A similar to B) $\Rightarrow$ (eigenvalues of A & B are same)
2. Suppose A has eigenvalues $\lambda_1, \dots \lambda_n$.
   What is the simplest matrix whose eigenvalues are $\lambda_1, \dots \lambda_n$?

$$B = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix} \quad \text{(diagonal).}$$

So, can we find matrix S that makes

$$B = S^{-1} A S \iff A = S B S^{-1} \ ?$$

Derivation Let A be 2×2 with 2 lin. independent eigenvectors:

$$A\underline{v_1} = \lambda_1 \underline{v_1}, \quad A\underline{v_2} = \lambda_2 \underline{v_2}$$

$$[A\underline{v_1}, A\underline{v_2}] = [\lambda_1 \underline{v_1}, \lambda_2 \underline{v_2}]$$

$$A \underbrace{[\underline{v_1}, \underline{v_2}]}_{V} = \underbrace{[\lambda_1 \underline{v_1}, \lambda_2 \underline{v_2}]}_{\text{want to see } V \text{ here}}$$

$$A V = [\lambda_1 \underline{v_1} + 0 \cdot \underline{v_2}, \ 0\underline{v_1} + \lambda_2 \underline{v_2}]$$

By Key Formula

$$A V = \left[\underbrace{[\underline{v_1}, \underline{v_2}]}_{V} \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix}, \underbrace{[\underline{v_1}, \underline{v_2}]}_{V} \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix} \right]$$

$$A V = V \left[\begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix} \right]$$

$$AV = V\underbrace{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}}_{D}$$

D = diagonal matrix with the eigenvalues on the diagonal!

$$AV = VD$$

Since V is nonsingular (why?), V^{-1} exists, $\Rightarrow$

$$\boxed{A = VDV^{-1}} \quad \text{or} \quad \boxed{D = V^{-1}AV}$$

where $V = [\underline{v}_1, \underline{v}_2]$ $\quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

lin. independent eigenvectors of A $\quad$ eigenvalues of A.

<u>Thus:</u> We've found a similarity transformation that <u>diagonalizes</u> A.

<u>Thm. 19</u> An $n \times n$ matrix A is diagonalizable <u>if and only if</u> it has n lin. independent eigenvectors $\underline{v}_1, \dots \underline{v}_n$:

$$A = V\,D\,V^{-1}$$

$[\underline{v}_1, \dots \underline{v}_n]$ $\quad$ $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

<u>lin. indep.</u> eigenvectors of A $\quad$ (some or all λ's may be repeated)

$\boxed{\text{MUST SEE: Ex. 1 in book}}$ ($A = VDV^{-1}$ with numbers).

!!! **Note:** Once you've found $V = [\underline{v}_1, \underline{v}_2]$, do **NOT** compute D as $V^{-1}AV$ (as the book does!), because $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ **always** (but you can make an error when computing $V^{-1}AV$, and <u>will</u> waste time).

<u>Interpretation of</u> $A = VDV^{-1}$.

Consider $A\underline{x}$, $\underline{x}$ is any vector in $\mathbb{R}^2$.

Recall what we did in Sec. 4.1 to find $A\underline{x}$.

In Sec. 4.1:

1) $\underline{x} = c_1\underline{v_1} + c_2\underline{v_2}$ (expand $\underline{x}$ over basis $\underline{v_1}, \underline{v_2}$).

How to find the coordinates c_1, c_2? See Sec. 3.4, topic 2 (Notes).

$$\underline{x} = \underbrace{[\underline{v_1}, \underline{v_2}]}_{V}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = V^{-1}\underline{x}.$$

2) $A\underline{x} = A c_1\underline{v_1} + A c_2\underline{v_2} = c_1 A\underline{v_1} + c_2 A\underline{v_2}$

$$= c_1\lambda_1\underline{v_1} + c_2\lambda_2\underline{v_2}.$$

<u>In this section:</u>

$$A\underline{x} = VD\underbrace{(V^{-1}\underline{x})}_{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}} = V\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= V\begin{pmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \end{pmatrix} = [\underline{v_1}, \underline{v_2}]\begin{pmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \end{pmatrix}$$

$$= (\lambda_1 c_1)\underline{v_1} + (\lambda_2 c_2)\underline{v_2}.$$

<u>Thus:</u>

$$A\underline{x} = V D \underline{V^{-1}\underline{x}}$$

- $V^{-1}\underline{x}$ → expand $\underline{x} = c_1\underline{v_1} + c_2\underline{v_2}$, or find coordinates $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ in basis $\underline{v_1}, \underline{v_2}$.
- D → stretch the coordinates: $\begin{pmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \end{pmatrix}$
- V → use the stretched coordinates to write the new expansion: $(\lambda_1 c_1)\underline{v_1} + (\lambda_2 c_2)\underline{v_2}$.

See the schematic picture of this process on p. 21-19 (end of Sec. 4.7-II).

Computing A^k

Let $A = VDV^{-1}$.

Then $A^2 = VD\underbrace{(V^{-1}\cdot V)}_{I}DV^{-1} = VDDV^{-1}$
$= VD^2V^{-1}$.

Similarly,
$$\boxed{A^k = VD^kV^{-1}}.$$

Q: Why is this useful?

A: Because D^k is very easy to compute, given that D is diagonal.

$$\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} = \begin{pmatrix}\lambda_1^2 & 0\\ 0 & \lambda_2^2\end{pmatrix},$$

$$\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}^k = \begin{pmatrix}\lambda_1^k & 0\\ 0 & \lambda_2^k\end{pmatrix}$$ ← Just raise the diagonal entries to k!

Thus, if
$$A = V\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}V^{-1} \Rightarrow$$
$$A^k = V\begin{pmatrix}\lambda_1^k & 0\\ 0 & \lambda_2^k\end{pmatrix}V^{-1}.$$

Must see Ex. 2 in book.

Note 1
$$\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}^k = \begin{pmatrix}\lambda_1^k & 0\\ 0 & \lambda_2^k\end{pmatrix}$$
only for diagonal matrices!
$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}^k \neq \begin{pmatrix}a^k & b^k\\ c^k & d^k\end{pmatrix} \text{ if } b, c \neq 0!$$

DO NOT make this common error!

Note 2

(Consistency check) Let: $A = V\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}V^{-1}$, then
$A^k = V\begin{pmatrix}\lambda_1^k & 0\\ 0 & \lambda_2^k\end{pmatrix}V^{-1}$.

So, since $A\underline{x} = \lambda_1 c_1 \underline{v_1} + \lambda_2 c_2 \underline{v_2}$, then
similarly $A^k \underline{x} = \lambda_1^k c_1 \underline{v_1} + \lambda_2^k c_2 \underline{v_2}$.
This is what we showed in Ex. 1 of Sec. 4.1.

<u>Conclusion to part 1</u> (for 2×2)

If A has 2 lin. indep. eigenvectors, then it is <u>diagonalizable</u>:

$$A = VDV^{-1}, \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

But there are also <u>non-diagonalizable</u> 2×2 matrices. They have only 1 eigenvector.
E.g., in Ex. 1 we showed that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is <u>non-diagonalizable</u>
(no S s.t. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = S \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} S^{-1}$.)

Non-diagonalizable matrices are <u>defective</u> matrices (Sec. 4.5) — they don't have enough eigenvectors to form a basis in $\mathbb{R}^2$ (or in $\mathbb{R}^n$ in general).

<u>Note:</u> "Non-diagonalizable" is <u>NOT related</u> at all to "non-singular"!

E.g., $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is singular but diagonalizable,
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is nonsingular but nondiagonalizable.

<u>HW for Part 1</u>: Sec. 4.7: 1, 2, 3, 4, 5, 7, 9, 10, 11, 25, 26, 27.