Project 1:

Examples of systems of linear equations

Goal

Practice setting up systems of linear equations.

General requirements

- You may work alone or with *one* other person. If you work with someone else, hand in *one* answer sheet with both of your names on it.
- No groups bigger than two. No collaboration between groups. **Please read 'My policies on Projects' posted on the course website.**
- Write your answers on the answer sheet provided in the last few pages of this document. Staple *with a staple, not a paper clip*[1] all the paper showing your *neatly presented*[2] work to the answer sheet.

Introduction

Systems of equations arise whenever the problem under consideration involves more than one variable and those variables are related to one another in some way. When these relations are (or can be approximated by) simple linear proportionalities, the systems of equations are linear. Linear systems arise in all areas of natural and social sciences.

In this project, you will encounter simple linear systems, involving only two or three variables, in planar geometry, in actuarial science, and in nutritional science. In each case, your main task will be to set up the linear system and then cast it in a matrix form. In the last two exercises, you will also be asked to use simple Matlab commands to obtain the solution of the systems you have set up.

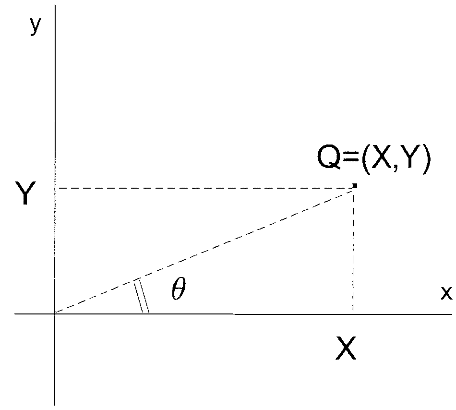
[1]Your grade will be reduced by 5% if you hand in a pile of non-stapled sheets.

[2]I will reduce your grade by an amount left to my discretion in each particular case if your work is presented in a messy way and I have to waste time deciphering it.

Exercise 1

Consider a point Q with coordinates (X, Y) in the xy-plane (see the figure below).

(a) Sketch a point Q_1 obtained from Q by the reflection about the x-axis. Let (X_1, Y_1) be the coordinates of Q_1. Next, in your sketch, label the coordinates of Q_1 in terms of X and Y (i.e., do *not* just label then X_1 and Y_1). Now, write down (**very simple**) equations that relate X_1 and Y_1 to X and Y. (*Hint*: Look at your sketch.) Rewrite these equations as a linear system of the form:



$$X_1 = t_{11}X + t_{12}Y, \quad Y_1 = t_{21}X + t_{22}Y,$$

where you are to determine the *numbers* t_{ij} from your equations.
Clarification: The t_{ij} are *constant* (i.e., independent of X and Y), and also *very simple*, numbers that make the above equations valid for *any* X and Y.

Finally, write this linear system in the matrix form

$$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = T_1 \begin{pmatrix} X \\ Y \end{pmatrix},$$

where your task is to determine matrix T_1 from the numbers t_{ij} that you have found.

(b) Repeat Part (a) for point (X_2, Y_2) obtained from point (X, Y) by the reflection about the y-axis.

(c) Repeat Part (a) for point (X_3, Y_3) obtained from point (X, Y) by the reflection about the line $y = x$.

(d) Repeat Part (a) for point $Q_4 = (X_4, Y_4)$ obtained from point (X, Y) by the rotation by angle ϕ (pronounced 'phi') in the counterclockwise direction about the origin. I suggest that you follow the steps outlined below, *making sketches as you proceed.*

Let r be the distance from point Q to the origin. Let θ be the angle between vector $\vec{OQ}$ and the positive x-axis. Similarly, let θ_4 be the angle between vector $\vec{OQ_4}$ and the positive x-axis. (Remember that you should be making sketches to help yourself visualize every step.) Clearly,

$$\theta_4 = \theta + \phi. \tag{1}$$

From your sketch you should see that

$$X = r\cos\theta, \qquad Y = r\sin\theta. \tag{2}$$

Now write equations similar to (2) for X_4, Y_4, and θ_4.
Next, write a system of linear equations that relates X_4 and Y_4 to X and Y. Proceed in two steps. First, use: (i) the equations you have just derived; (ii) Equation (1), (iii) the formulae

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta \\ \sin(\alpha+\beta) &= \cos\alpha\,\sin\beta + \sin\alpha\,\cos\beta, \end{aligned} \tag{3}$$

and multiply out the resulting expression. (Do this for X_4 first and worry about Y_4 later.) Second, look at your expression and notice in it expressions for X ad Y as defined by Equations (2). Once you have written X_4 as a linear combination of X and Y, repeat this for Y_4.
Finally, write the linear system for X_4 and Y_4 in matrix form:

$$\begin{pmatrix} X_4 \\ Y_4 \end{pmatrix} = T_4 \begin{pmatrix} X \\ Y \end{pmatrix},$$

where matrix T_4 contains as its entries only $\cos\phi$ and $\sin\phi$.

<u>Extra credit</u> (11%; partial credit is given only if the solution is mostly correct)

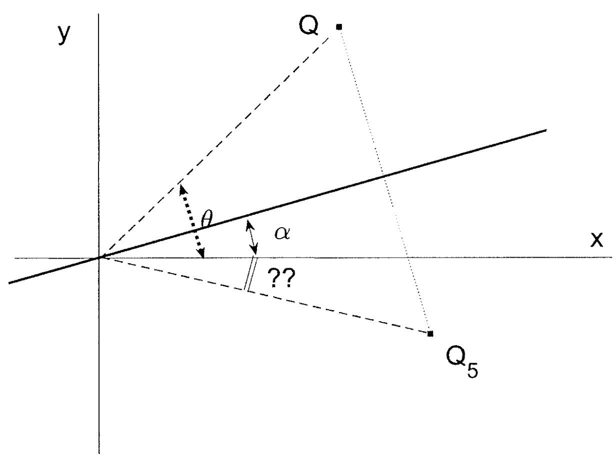
Repeat Part (a) for point (X_5, Y_5) obtained from point (X, Y) by the reflection about the line $y = x \cdot \tan\alpha$.

Hint: See the picture to the right.

Note 1: The sense of rotation determines the sign of the angle: counterclockwise is positive and clockwise is negative. For example, in the picture to the right, α and θ are positive and the angle marked as "??" is negative.
(To check your understanding: does the answer to part (d) depends on the sense of rotation?)

Note 2: After you obtain an answer to this Extra Credit, check if it agrees with answer(s) to one or more regular parts of this Exercise.

Note 3: You **must explain** every step of your solution, especially that/those alluded to by the previous Notes.



Exercise 2

By reviewing its donation records, the alumni office of a college finds that 80% of its alumni who contribute to the alumni fund in any given year also contribute the next year, and that 30% of those who do not contribute one year will contribute the next. Denote

$$\underline{\mathbf{x}}^{(1)} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix}$$

to be the numbers of alumni who, respectively, did and did not contribute in year 1.[3] Similarly, denote $\underline{\mathbf{x}}^{(2)} = (x_1^{(2)}, x_2^{(2)})^T$ to be the numbers of alumni who did and did not contribute in year 2.

[3] A cosmetic notation: In order not to use up a few lines of text to display a single formula, as above, we will use the "transpose notation" defined on p. 63 of the textbook, using the superscript 'T': $\underline{\mathbf{x}}^{(1)} = (x_1^{(1)}, x_2^{(1)})^T$. This shorter notation is equivalent to the longer, "full-display" notation used above. If you have any questions about it, you should consult p. 63 of the textbook.

(a) Write a linear system of equations that relates $x_1^{(2)}$ and $x_2^{(2)}$ to $x_1^{(1)}$ and $x_2^{(1)}$. It is recommended that you review how this was done in Example 1 in the posted notes for Section 1.1.

(b) Rewrite this system in matrix form:

$$\begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \end{pmatrix} = P_1 \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix}, \tag{4}$$

where your task is to determine the matrix P_1, called the *transition* matrix.

(c) Let $\underline{\mathbf{x}}^{(3)} = (x_1^{(3)}, x_2^{(3)})^T$ be the numbers of alumni who did and did not contribute to the alumni fund in the following year (year 3). Start by writing down a matrix equation, analogous to Equation (4), that relates $\underline{\mathbf{x}}^{(3)}$ to $\underline{\mathbf{x}}^{(2)}$. Then, use the associative property of matrix multiplication to obtain the relation between $\underline{\mathbf{x}}^{(3)}$ and $\underline{\mathbf{x}}^{(1)}$ in the form:

$$\underline{\mathbf{x}}^{(3)} = P_2\,\underline{\mathbf{x}}^{(1)}, \tag{5}$$

where your task is to determine how matrix P_2 is related to P_1 (see Equation (4)).

(d) Extend the results of Part (c) to later years. Namely, compute how many people contributed in years 4 and 5. That is, step by step: First, similarly to Equation (5), write a relation $\underline{\mathbf{x}}^{(4)} = P_3\,\underline{\mathbf{x}}^{(1)}$, where you need to *explicitly* use the associative property to relate P_3 to P_1. Then repeat this for $\underline{\mathbf{x}}^{(5)} = P_4\,\underline{\mathbf{x}}^{(1)}$. At this point (if not earlier), you probably notice a pattern of how P_N is related to P_1. Finally, state this relation and then write an equation similar to Equation (5) that relates $\underline{\mathbf{x}}^{(N)}$ to $\underline{\mathbf{x}}^{(1)}$.

(e) Suppose $\underline{\mathbf{x}}^{(1)} = (0, 1000)^T$ (i.e., in the first year, no one contributed to the alumni fund). Use the pattern that you established in the previous part to record x_1 and x_2 in years 2 through 10 and then in year 15 in a Table. State your observation(s) from this Table.

Technical note: To avoid doing many multiplications of a vector by a matrix by hand, use Matlab. Read Sec. A.2 in Appendix A in the textbook about how to enter a matrix and a vector. To multiply matrix A by vector $\underline{\mathbf{x}}$, enter `A*x` (see Sec. A.7).

Exercise 3

In the previous Exercises, your approach to writing down a linear system has been as follows: You related *each of* certain quantities q_1 and q_2 to two other quantities, p_1 and p_2 (e.g., in Exercise 2a, $q_1 = x_1^{(2)}$, $q_2 = x_2^{(2)}$ and $p_1 = x_1^{(1)}$, $p_2 = x_2^{(1)}$). In this Exercise, you will be asked to follow an alternative approach, based on *the formula which should be prominently displayed on the front cover of your notebook.* [4]

Diets, i.e. nutritional programs prescribed by nutrition specialists, are based on the idea that a person needs to consume a certain amount of key nutrients, such as protein, fat, carbohydrates, vitamins, minerals, etc. in correct proportions. To achieve the desired amounts and proportions of nutrients, one has to incorporate a large variety of ingredients in the diet. Each ingredient supplies several of the required nutrients, but not in the correct proportion. The following example illustrates the problem on a small scale. The data shown in the Table below refer to the Cambridge Diet, a popular weight-loss diet in the 1980s.

[4] In the textbook, this is Equation (8) in Sec. 1.5 and also the first equation for $A\mathbf{x}$ in Theorem 5.

Amounts [g] of nutrients supplied per 100 g of ingredient

Nutrient	Nonfat milk	Soy flour	Whey	Amount [grams per day] required by Cambridge Diet
Protein	36	51	13	33
Carbohydrates	52	34	74	45
Fat	0	7	1	3

(a) Denote

$$\underline{\mathbf{a}}_1 = \begin{pmatrix} 36 \\ 52 \\ 0 \end{pmatrix}, \quad \underline{\mathbf{a}}_2 = \begin{pmatrix} 51 \\ 34 \\ 7 \end{pmatrix}, \quad \underline{\mathbf{a}}_3 = \begin{pmatrix} 13 \\ 74 \\ 1 \end{pmatrix}, \quad \underline{\mathbf{b}} = \begin{pmatrix} 33 \\ 45 \\ 3 \end{pmatrix}$$

to be the nutrient contents of one unit (i.e., 100 g) of nonfat milk, soy flour, whey, and the Cambridge Diet, respectively (see the Table above). Let x_1, x_2, x_3 be the numbers of 100-gram units of nonfat milk, soy flour, and whey that are to be used to obtain one 100-gram unit of this diet. (These numbers are yet unknown, but you will be asked to find them below.)
What are the meanings of each of $x_1\underline{\mathbf{a}}_1$, $x_2\underline{\mathbf{a}}_2$, $x_3\underline{\mathbf{a}}_3$?

Hint: If you have difficulty answering this question, try to proceed in steps. First, write down[5] the meanings of $\underline{\mathbf{a}}_1$ and $\underline{\mathbf{b}}$, as listed above. Note that one of them refers to the Cambridge Diet and the other does not. Second, write the meaning of x_1. Hopefully, these steps will help you recognize the meaning of $x_1\,\underline{\mathbf{a}}_1$.

(b) Use your answer in Part (a) to write down a vector equation that can be used to find x_1, x_2, x_3. (On each side of this equation there will be either a vector or a sum of vectors.)

(c) Rewrite this vector equation in matrix form $A\underline{\mathbf{x}} = \underline{\mathbf{y}}$, where your task is to identify the matrix A and the vector $\underline{\mathbf{y}}$.

(d) Solve this matrix equation using the `rref` command of Matlab (see Secs. A.2 and A.3 in Appendix A of the textbook).

Extra Credit (15%; partial credit is given only if the solution is mostly correct)

You may notice that the amounts of milk, flour, and whey in the diet do not add up to 100 g, as one might expect at first. Upon some pondering, one may surmise that perhaps they should add up to 81 g, which is the combined amount of protein, carbohydrates, and fat in the diet. But, again, they do not. Figure out why. Then modify the statement of the problem so that the amounts obtained as its new solutions would add up to 100 g. In doing so, you must **not** change any of the numbers that are already present in the Table in Exercise 3.

Note: It is possible that you may figure out the problem and propose a way to fix it just using common sense. However, we will study a theory underlying the concepts involved in Chapter 3; you will also encounter a related situation in Project 3. Therefore, even if you don't figure out this problem here, you are welcome to revisit it after completing Project 3. I will still give extra credit for a correct solution; just not 15% as now, but only 9%. I will remind you of this opportunity when time comes.

[5] For yourself, not on the Answer sheet.

Acknowledgment:

Exercises 2 and 3 in this project are based on the material found in:

C. Rorres and H. Anton, *Applications of Linear Algebra*, 2nd ed., Wiley (New York, 1977) and

D.C. Lay, *Linear Algebra and Its Applications*, 3rd ed., Pearson/Addison Wesley (Boston, 2006).

Reminder: ***Staple to this sheet all the additional pages with your neatly presented work, including Matlab printouts.***

Name(s): ______________________________

Exercise 1

(a) *(7 points)*: On the right, sketch Q, Q_1 and label the coordinates of Q_1, expressing these coordinates in terms of X and/or Y in your sketch.
(E.g., if you have found that $X_1 = 2X$ and $Y_1 = 3Y$, your label should be: $Q_1(2X, 3Y)$.)

System of equations:

This system in the matrix form:

(b) *(7 points)*: On the right, sketch Q, Q_2 and mark the coordinates of Q_2, expressing them in terms of X and Y.

System of equations:

This system in the matrix form:

Exercise 1, continued

(c) *(7 points)*: On the right, sketch Q, Q_3 and mark the coordinates of Q_3, expressing them in terms of X and Y.

System of equations:

This system in the matrix form:

(d) *(11 points)*: Equations similar to (2) for X_4, Y_4, and θ_4:

The system referred to after Equation (3) (attach extra pages showing your work):

This system in the matrix form:

Extra credit: Write the answer in the matrix form here and attach additional pages with your work.

Exercise 2

(a) *(6 points)*: System of equations:

(b) *(6 points)*: This system in matrix form:

(c) *(7 points)*: Equation relating $\underline{\mathbf{x}}^{(3)}$ to $\underline{\mathbf{x}}^{(2)}$:

Equation relating $\underline{\mathbf{x}}^{(3)}$ to $\underline{\mathbf{x}}^{(1)}$ in terms of matrix P_1 (yes, P_1, not P_2):

Relation between matrices P_2 and P_1 (provide a derivation based on Theorem 8, showing all steps):

(d) *(7 points)*: On a separate page, write the derivations (again, explicitly using Theorem 8) of equations relating $\underline{\mathbf{x}}^{(4)}$ and $\underline{\mathbf{x}}^{(5)}$ to $\underline{\mathbf{x}}^{(1)}$. Based on the observed pattern, state an equation relating $\underline{\mathbf{x}}^{(N)}$ to $\underline{\mathbf{x}}^{(1)}$ for any N:

(e) *(8 points)*: Fill the Table below. Since you record numbers of people, round them to integers.

N	1	2	3	4	5	6	7	8	9	10	15
$x_1^{(N)}$	0										
$x_2^{(N)}$	1000										

Attach a printout showing how you obtained numbers in this Table. (If you did the calculations by hand or using a pocket calculator (either of which methods being *discouraged*), you must explain what you did and show all steps of your work on separate pages.)

State your observation from the above table:

Exercise 3

(a) *(9 points)*: What is the meaning of $x_1\underline{\mathbf{a}}_1$? (Since the meanings of $x_2\underline{\mathbf{a}}_2$ and $x_3\underline{\mathbf{a}}_3$ are analogous, you do not need to state them.) *Note*: Your answer must be in *layman* terms and *not* use *any* Linear Algebra. For example, you must *not* refer to x_1 in your answer.

(b) *(8 points)*: Vector equation involving x_1, x_2, x_3:

(c) *(7 points)*: This equation in matrix form:

(d) *(10 points)*: Solution of this linear system (attach a printout if you used a software):

Amounts of the listed ingredients, in grams, required for one 100-gram unit of the Cambridge Diet:

Attach your work for the Extra Credit for this Exercise, if you got it.