(a) Let $p(t)$ satisfy (4). Show that

$$\frac{d^2p}{dt^2}=(N-p)(cp+c')(cN-2cp-c').$$

(b) Show that $p(t)$ has a point of inflection, at which $dp/dt$ achieves a maximum, if, and only if, $c'/c<N$.
(c) Assume that $p(t)$ has a point of inflection at $t=t^*$. Show that $p(t^*)\leqslant N/2$.

**4.** Solve the initial-value problem (7).

**5.** It seems reasonable to take the time span between the date when 20% of the firms had introduced the innovation and the date when 80% of the firms had introduced the innovation, as the rate of imitation.
(a) Show from our model that this time span is $4(\ln 2)/k$.
(b) For each of the twelve innovations, compute this time span from the data in Table 1, and compare with the observed value in Figure 3.

**6.** (a) Show from our model that $(1/k)\ln(n-1)$ years elapse before 50% of the firms introduce an innovation.
(b) Compute this time span for each of the 12 innovations and compare with the observed values in Figure 3.

## 1.7 An atomic waste disposal problem

For several years the Atomic Energy Commission (now known as the Nuclear Regulatory Commission) had disposed of concentrated radioactive waste material by placing it in tightly sealed drums which were then dumped at sea in fifty fathoms (300 feet) of water. When concerned ecologists and scientists questioned this practice, they were assured by the A.E.C. that the drums would never develop leaks. Exhaustive tests on the drums proved the A.E.C. right. However, several engineers then raised the question of whether the drums could crack from the impact of hitting the ocean floor. "Never," said the A.E.C. "We'll see about that," said the engineers. After performing numerous experiments, the engineers found that the drums could crack on impact if their velocity exceeded forty feet per second. The problem before us, therefore, is to compute the velocity of the drums upon impact with the ocean floor. To this end, we digress briefly to study elementary Newtonian mechanics.

Newtonian mechanics is the study of Newton's famous laws of motion and their consequences. Newton's first law of motion states that an object will remain at rest, or move with constant velocity, if no force is acting on it. A force should be thought of as a push or pull. This push or pull can be exerted directly by something in contact with the object, or it can be exerted indirectly, as the earth's pull of gravity is.

Newton's second law of motion is concerned with describing the motion of an object which is acted upon by several forces. Let $y(t)$ denote the position of the center of gravity of the object. (We assume that the object moves in only one direction.) Those forces acting on the object, which tend to increase $y$, are considered positive, while those forces tending to decrease $y$ are considered negative. The resultant force $F$ acting on an object is defined to be the sum of all positive forces minus the sum of all negative forces. Newton's second law of motion states that the acceleration $d^2y/dt^2$ of an object is proportional to the resultant force $F$ acting on it; i.e.,

$$\frac{d^2y}{dt^2}=\frac{1}{m}F. \tag{1}$$

The constant $m$ is the mass of the object. It is related to the weight $W$ of the object by the relation $W=mg$, where $g$ is the acceleration of gravity. Unless otherwise stated, we assume that the weight of an object and the acceleration of gravity are constant. We will also adopt the English system of units, so that $t$ is measured in seconds, $y$ is measured in feet, and $F$ is measured in pounds. The units of $m$ are then slugs, and the gravitational acceleration $g$ equals 32.2 ft/s$^2$.

**Remark.** We would prefer to use the mks system of units, where $y$ is measured in meters and $F$ is measured in newtons. The units of $m$ are then kilograms, and the gravitational acceleration equals 9.8 m/s$^2$. In the third edition of this text, we have changed from the English system of units to the mks system in Section 2.6. However, changing to the mks system in this system would have caused undue confusion to the users of the first and second editions. This is because of the truncation error involved in converting from feet to meters and pounds to newtons.

We return now to our atomic waste disposal problem. As a drum descends through the water, it is acted upon by three forces $W$, $B$, and $D$. The force $W$ is the weight of the drum pulling it down, and in magnitude, $W=527.436$ lb. The force $B$ is the buoyancy force of the water acting on the drum. This force pushes the drum up, and its magnitude is the weight of the water displaced by the drum. Now, the Atomic Energy Commission used 55 gallon drums, whose volume is 7.35 ft$^3$. The weight of one cubic foot of salt water is 63.99 lb. Hence $B=(63.99)(7.35)=470.327$ lb.

The force $D$ is the drag force of the water acting on the drum; it resists the motion of the drum through the water. Experiments have shown that any medium such as water, oil, and air resists the motion of an object through it. This resisting force acts in the direction opposite the motion, and is usually directly proportional to the velocity $V$ of the object. Thus, $D=cV$, for some positive constant $c$. Notice that the drag force increases as $V$ increases, and decreases as $V$ decreases. To calculate $D$, the engineers conducted numerous towing experiments. They concluded that the orientation of the drum had little effect on the drag force, and that

$$D=0.08\,V\frac{(\text{lb})(\text{s})}{\text{ft}}.$$

Now, set $y=0$ at sea level, and let the direction of increasing $y$ be downwards. Then, $W$ is a positive force, and $B$ and $D$ are negative forces. Consequently, from (1),

$$\frac{d^2y}{dt^2}=\frac{1}{m}(W-B-cV)=\frac{g}{W}(W-B-cV).$$

We can rewrite this equation as a first-order linear differential equation for $V=dy/dt$; i.e.,

$$\frac{dV}{dt}+\frac{cg}{W}V=\frac{g}{W}(W-B). \tag{2}$$

Initially, when the drum is released in the ocean, its velocity is zero. Thus, $V(t)$, the velocity of the drum, satisfies the initial-value problem

$$\frac{dV}{dt}+\frac{cg}{W}V=\frac{g}{W}(W-B), \qquad V(0)=0, \tag{3}$$

and this implies that

$$V(t)=\frac{W-B}{c}\left[1-e^{(-cg/W)t}\right]. \tag{4}$$

Equation (4) expresses the velocity of the drum as a function of time. In order to determine the impact velocity of the drum, we must compute the time $t$ at which the drum hits the ocean floor. Unfortunately, though, it is impossible to find $t$ as an explicit function of $y$ (see Exercise 2). Therefore, we cannot use Equation (4) to find the velocity of the drum when it hits the ocean floor. However, the A.E.C. can use this equation to try and prove that the drums do not crack on impact. To wit, observe from (4) that $V(t)$ is a monotonic increasing function of time which approaches the limiting value

$$V_T=\frac{W-B}{c}$$

as $t$ approaches infinity. The quantity $V_T$ is called the terminal velocity of the drum. Clearly, $V(t)\leqslant V_T$, so that the velocity of the drum when it hits the ocean floor is certainly less than $(W-B)/c$. Now, if this terminal velocity is less than 40 ft/s, then the drums could not possibly break on impact. However,

$$\frac{W-B}{c}=\frac{527.436-470.327}{0.08}=713.86\text{ ft/s},$$

and this is way too large.

It should be clear now that the only way we can resolve the dispute between the A.E.C. and the engineers is to find $v(y)$, the velocity of the drum as a function of position. The function $v(y)$ is very different from the function $V(t)$, which is the velocity of the drum as a function of time. However, these two functions are related through the equation

$$V(t)=v(y(t))$$

if we express $y$ as a function of $t$. By the chain rule of differentiation, $dV/dt=(dv/dy)(dy/dt)$. Hence

$$\frac{W}{g}\frac{dv}{dy}\frac{dy}{dt}=W-B-cV.$$

But $dy/dt=V(t)=v(y(t))$. Thus, suppressing the dependence of $y$ on $t$, we see that $v(y)$ satisfies the first-order differential equation

$$\frac{W}{g}v\frac{dv}{dy}=W-B-cv, \quad \text{or} \quad \frac{v}{W-B-cv}\frac{dv}{dy}=\frac{g}{W}.$$

Moreover,

$$v(0)=v(y(0))=V(0)=0.$$

Hence,

$$\int_0^v \frac{r\,dr}{W-B-cr}=\int_0^y \frac{g}{W}ds=\frac{gy}{W}.$$

Now,

$$\int_0^v \frac{r\,dr}{W-B-cr}=\int_0^v \frac{r-(W-B)/c}{W-B-cr}dr+\frac{W-B}{c}\int_0^v \frac{dr}{W-B-cr}$$

$$=-\frac{1}{c}\int_0^v dr+\frac{W-B}{c}\int_0^v \frac{dr}{W-B-cr}$$

$$=-\frac{v}{c}-\frac{(W-B)}{c^2}\ln\frac{|W-B-cv|}{W-B}.$$

We know already that $v<(W-B)/c$. Consequently, $W-B-cv$ is always positive, and

$$\frac{gy}{W}=-\frac{v}{c}-\frac{(W-B)}{c^2}\ln\frac{W-B-cv}{W-B}. \tag{5}$$

At this point, we are ready to scream in despair since we cannot find $v$ as an explicit function of $y$ from (5). This is not an insurmountable difficulty, though. As we show in Section 1.11, it is quite simple, with the aid of a digital computer, to find $v(300)$ from (5). We need only supply the computer with a good approximation of $v(300)$ and this is obtained in the following manner. The velocity $v(y)$ of the drum satisfies the initial-value problem

$$\frac{W}{g}v\frac{dv}{dy}=W-B-cv, \qquad v(0)=0. \tag{6}$$

Let us, for the moment, set $c=0$ in (6) to obtain the new initial-value problem

$$\frac{W}{g}u\frac{du}{dy}=W-B, \qquad u(0)=0. \tag{6'}$$

(We have replaced $v$ by $u$ to avoid confusion later.) We can integrate (6′) immediately to obtain that

$$\frac{W}{g}\frac{u^2}{2}=(W-B)y, \quad \text{or} \quad u(y)=\left[\frac{2g}{W}(W-B)y\right]^{1/2}.$$

In particular,

$$u(300)=\left[\frac{2g}{W}(W-B)300\right]^{1/2}=\left[\frac{2(32.2)(57.109)(300)}{527.436}\right]^{1/2}$$
$$\cong\sqrt{2092}\cong 45.7 \text{ ft/s}.$$

We claim, now, that $u(300)$ is a very good approximation of $v(300)$. The proof of this is as follows. First, observe that the velocity of the drum is always greater if there is no drag force opposing the motion. Hence,

$$v(300)<u(300).$$

Second, the velocity $v$ increases as $y$ increases, so that $v(y)\leqslant v(300)$ for $y\leqslant 300$. Consequently, the drag force $D$ of the water acting on the drum is always less than $0.08\times u(300)\cong 3.7$ lb. Now, the resultant force $W-B$ pulling the drum down is approximately 57.1 lb, which is very large compared to $D$. It stands to reason, therefore, that $u(y)$ should be a very good approximation of $v(y)$. And indeed, this is the case, since we find numerically (see Section 1.11) that $v(300)=45.1$ ft/s. Thus, the drums can break upon impact, and the engineers were right.

*Epilog.* The rules of the Atomic Energy Commission now expressly forbid the dumping of low level atomic waste at sea. This author is uncertain though, as to whether Western Europe has also forbidden this practice.

**Remark.** The methods introduced in this section can also be used to find the velocity of any object which is moving through a medium that resists the motion. We just disregard the buoyancy force if the medium is not water. For example, let $V(t)$ denote the velocity of a parachutist falling to earth under the influence of gravity. Then,

$$\frac{W}{g}\frac{dV}{dt}=W-D$$

where $W$ is the weight of the man and the parachute, and $D$ is the drag force exerted by the atmosphere on the falling parachutist. The drag force on a bluff object in air, or in any fluid of small viscosity is usually very nearly proportional to $V^2$. Proportionality to $V$ is the exceptional case, and occurs only at very low speeds. The criterion as to whether the square or the linear law applies is the "Reynolds number"

$$R=\rho VL/\mu.$$

$L$ is a representative length dimension of the object, and $\rho$ and $\mu$ are the density and viscosity of the fluid. If $R<10$, then $D\sim V$, and if $R>10^3$, $D\sim V^2$. For $10<R<10^3$, neither law is accurate.