lution of the nonhomogeneous equation (namely the constant function $y = 2$) and the general solution of the homogeneous equation (namely $y = Ce^{-t^2}$). Note that the initial condition was imposed on the general solution as the last step. This will always be the case.

Discontinuous Coefficient Functions

In some applications, physical conditions undergo abrupt changes. For example, a hot metal object might be plunged suddenly into a cooling bath, or we might throw a switch and abruptly change the source voltage in an electrical network.

Such applications are often modeled by an initial value problem

$$y' + p(t)y = g(t), \qquad y(a) = y_0, \qquad a \le t \le b,$$

where one or both of the functions $p(t)$ and $g(t)$ have a jump discontinuity at some point, say $t = c$, where $a < c < b$. In such cases, even though $y'(t)$ is not continuous at $t = c$, we expect on physical grounds that the solution $y(t)$ is continuous at $t = c$. For these problems we first solve the initial value problem on the interval $a \le t < c$; the solution $y(t)$ will have a one-sided limit,

$$\lim_{t \to c^-} y(t) = y(c-).$$

To complete the solution, we use the limiting value $y(c-)$ as the initial condition on the subinterval $[c, b]$ and then solve a second initial value problem on $[c, b]$.

EXAMPLE 5

Solve the following initial value problem on the interval $0 \le t \le 2$:

$$y' - y = g(t), \qquad y(0) = 0, \qquad \text{where } g(t) = \begin{cases} 1, & 0 \le t < 1 \\ -2, & 1 \le t \le 2. \end{cases}$$

Solution: The graph of $g(t)$ is shown in Figure 2.4(*a*); it has a jump discontinuity at $t = 1$. On the interval $[0, 1)$, the differential equation reduces to $y' - y = 1$. The general solution is

$$y(t) = Ce^t - 1.$$

Imposing the initial condition, we obtain $y(t) = e^t - 1, 0 \le t < 1$. As t approaches 1 from the left, $y(t)$ approaches the value $e - 1$. Therefore, to complete the solution process, we solve a second initial value problem,

$$y' - y = -2, \qquad y(1) = e - 1, \qquad 1 \le t \le 2.$$

The solution of this initial value problem is

$$y(t) = \left(1 - \frac{3}{e}\right) e^t + 2, \qquad 1 \le t \le 2.$$

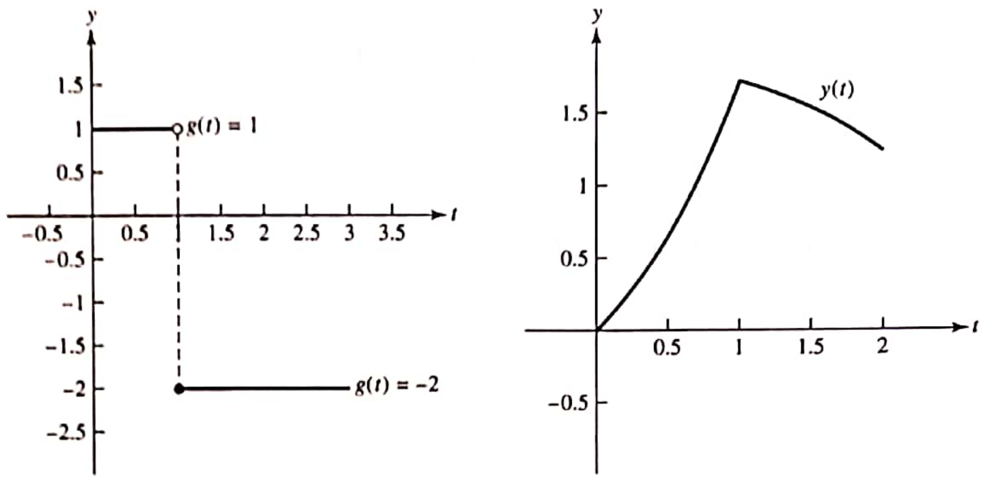
Combining the individual solutions of these two initial value problems, we obtain the solution for the entire interval $0 \le t \le 2$:

$$y(t) = \begin{cases} e^t - 1, & 0 \le t < 1 \\ \left(1 - \dfrac{3}{e}\right) e^t + 2, & 1 \le t \le 2. \end{cases}$$

(continued)

(continued)

The graph of $y(t)$ is shown in Figure 2.4(b). Note that $y(t)$ is continuous on the entire t-interval of interest. However, $y(t)$ is not differentiable at $t = 1$.



(a) (b)

FIGURE 2.4

(a) The coefficient function $g(t)$ of the differential equation $y' - y = g(t)$ in Example 5 has a jump discontinuity at $t = 1$. (b) The solution of $y' - y = g(t)$, $y(0) = 0$ is continuous on the interval $0 \le t \le 2$, but is not differentiable at $t = 1$. ❖

EXERCISES

Exercises 1–10:

For each initial value problem,

(a) Find the general solution of the differential equation.

(b) Impose the initial condition to obtain the solution of the initial value problem.

1. $y' + 3y = 0, \quad y(0) = -3$

2. $2y' - y = 0, \quad y(-1) = 2$

3. $2ty - y' = 0, \quad y(1) = 3$

4. $ty' - 4y = 0, \quad y(1) = 1$

5. $y' - 3y = 6, \quad y(0) = 1$

6. $y' - 2y = e^{3t}, \quad y(0) = 3$

7. $2y' + 3y = e^t, \quad y(0) = 0$

8. $y' + y = 1 + 2e^{-t}\cos 2t, \quad y(\pi/2) = 0$

9. $2y' + (\cos t)y = -3\cos t, \quad y(0) = -4$

10. $y' + 2y = e^{-t} + t + 1, \quad y(-1) = e$

Exercises 11–24:

Find the general solution.

11. $ty' + 4y = 0$

12. $y' + (1 + \sin t)y = 0$

13. $y' - 2(\cos 2t)y = 0$

14. $(t^2 + 1)y' + 2ty = 0$

15. $\dfrac{y'}{(t^2 + 1)y} = 3$

16. $y + e^t y' = 0$

17. $y' + 2y = 1$

18. $y' + 2y = e^{-t}$

19. $y' + 2y = e^{-2t}$

20. $y' + 2ty = t$

21. $ty' + 2y = t^2, \quad t > 0$

22. $(t^2 + 4)y' + 2ty = t^2(t^2 + 4)$