Lecture 10. Introduction to 2nd-order DEs: motivation, basic properties, linear DEs.

(1) Definition

$$y'' = F(t, y, y') \qquad (1)$$

is called a 2nd-order DE (shorthand notation: DE-2, as opposed to the 1st-order DEs, or DE-1).

If F is a nonlinear function of y and/or y', then DE-2 can be solved analytically (i.e. not numerically by a computer) only in very few special cases (even much more rarely than a nonlinear DE-1). E.g., $y'' = f(y)$ (an autonomous eq.) cannot be solved except for some very special $f(y)$.

Then we look at the linear DE-2:

We'll study only this DE in Chap. 3

$$y'' + p(t)y' + q(t)y = g(t). \qquad (2)$$

Unlike linear DE-1, Eq. (2) cannot be solved for general $p(t)$ & $q(t)$. This is a big difference between linear DE-1 & DE-2. We'll show it later in this lecture.

In general, solving a DE-2 is much harder than a DE-1.

However, there are also some similarities in their properties.

2 Motivation

Q: Why consider DE-2?

A: Most processes in Mechanics are governed by the 2nd Law of Newton:

$$m\vec{a} = \underbrace{\sum \vec{F}}_{\text{vector sum of all forces}}$$

acceleration

In one dimension, $\vec{a} = \frac{d^2y}{dt^2}$, $\Rightarrow$

$$m y'' = F(t, y, y') \qquad (3)$$

position velocity

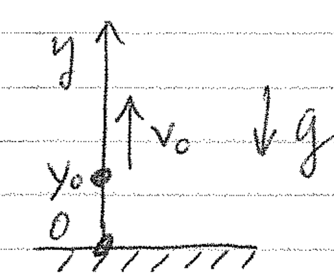
which has the form (1).

3 What to expect of the solution?

Consider a very simple DE-2 (HS Physics or Calc. III):

$$m y'' = -mg \quad \text{gravity}$$

$$\begin{cases} y'' = -g \\ y(0) = y_0, \; y'(0) = v_0 . \end{cases}$$



Solution: 1) $y'' = -g \Rightarrow y' = -\int g\,dt = -gt + C_1$

$$y'(0) = v_0 = -g \cdot 0 + C_1 \Rightarrow C_1 = v_0 .$$

2) $y' = -gt + v_0 \Rightarrow y = \int(-gt+v_0)dt$
$= -gt^2/2 + v_0 t + C_2$.

$y(0) = y_0 = -g\cdot 0 + v_0 \cdot 0 + C_2 \Rightarrow C_2 = y_0$.

Thus $y(t) = -\frac{gt^2}{2} + v_0 t + y_0$

Observations: 1) We have two constants, C_1 & C_2, not one as for DE-1.

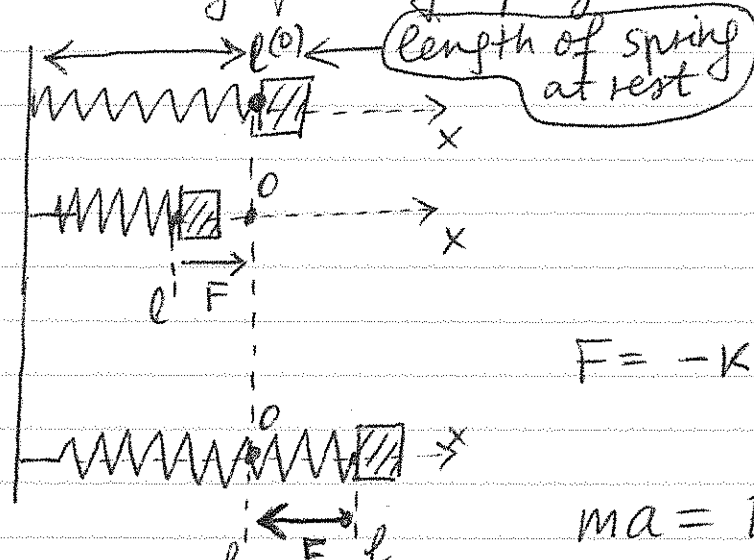
2) We need two initial cond's, $y(0)$ & $y'(0)$, to determine these constants (although, in general, they depend on $y(0)$, $y'(0)$ in a more complicated way).

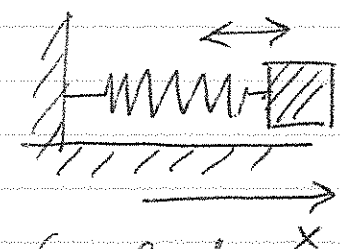
Meanings: $y(0)$ = initial position; $y'(0)$ = initial velocity.

4 A fundamental model: linear oscillator

Consider the motion of a mass on a spring:

1) Restoring force of spring





(neglect friction)

$F = -k(\underbrace{l - l_0}_{x})$;

$ma = F \Rightarrow$

$$mx'' = -kx$$

$$mx'' + kx = 0$$

$$x'' + \underbrace{\frac{k}{m}}_{\omega^2} x = 0 \quad \leftarrow \text{"omega square"}$$

$$\boxed{x'' + \omega^2 x = 0} \leftarrow (4)$$

Since mass on a spring oscillates, Eq. (4) is called the <u>linear oscillator model</u>.

It is fundamental to DE-2s, just like $y' = ay$ is fundamental for DE-1.

2) <u>General sol'n</u> (to be derived later, in Sec. 3.5)

a) $x_1 = \cos\omega t$ is a sol'n of (4).

<u>Check:</u> $x_1' = -\omega \sin\omega t$

$x_1'' = -\omega \cdot \omega \cdot \cos\omega t = -\omega^2 \cos\omega t$.

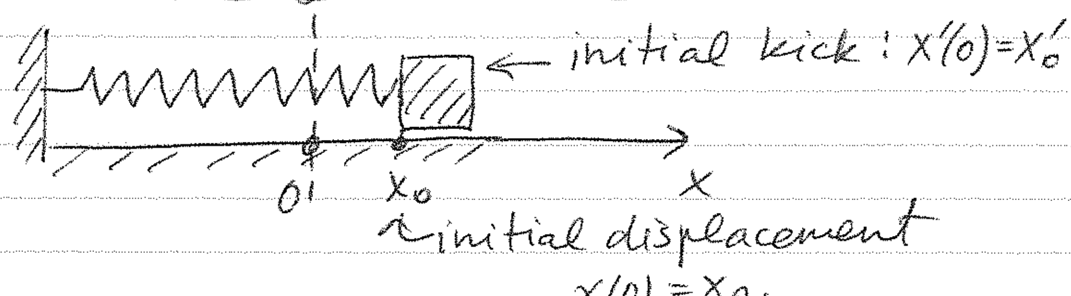
$x_1'' + \omega^2 x_1 = -\omega^2 \cos\omega t + \omega^2 \cdot \cos\omega t = 0$. ✓.

b) $x_2 = \sin\omega t$ is also a sol'n of (4)
(proof: @ home)

c) $x = c_1 x_1 + c_2 x_2 = c_1 \cos\omega t + c_2 \sin\omega t$; $c_{1,2} = \text{const}$
is also a solution of (4) (proof: @ home).

<u>Note:</u> Looks like the superposition principle for DE-1.

3) Solution for given init. cond's.



Then:

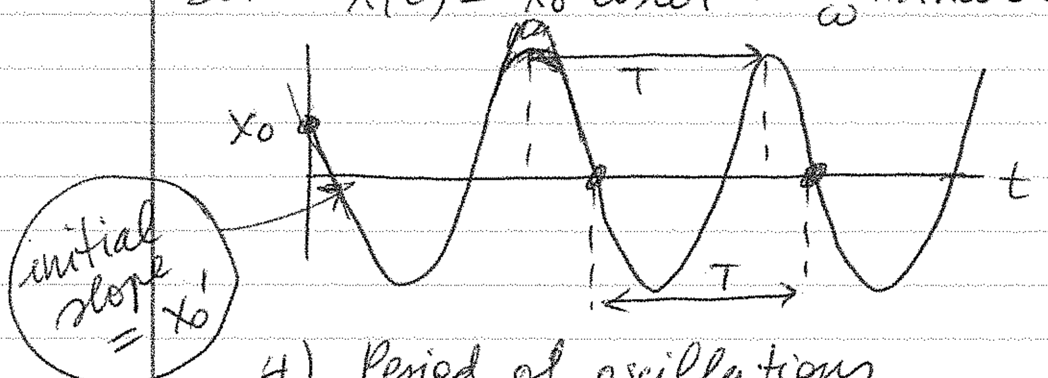
$$x(0) = c_1\cdot\underbrace{\cos(\omega\cdot 0)}_{1} + c_2\cdot\underbrace{\sin(\omega\cdot 0)}_{0} = x_0$$

$$x'(0) = c_1(\cos\omega t)'\Big|_{t=0} + c_2(\sin\omega t)'\Big|_{t=0}$$

$$= -c_1\cdot\omega\cdot\underbrace{\sin(\omega\cdot 0)}_{0} + c_2\cdot\omega\cdot\underbrace{\cos(\omega\cdot 0)}_{1}$$

$$\Rightarrow \begin{cases} c_1\cdot 1 + c_2\cdot 0 = x_0 \\ c_1\cdot 0 + c_2\cdot\omega = x_0' \end{cases} \Rightarrow \begin{cases} c_1 = x_0 \\ c_2 = x_0'/\omega. \end{cases}$$

So: $$x(t) = x_0\cdot\cos\omega t + \frac{x_0'}{\omega}\cdot\sin\omega t.$$



4) Period of oscillations

Note: $\boxed{\cos(\omega t_1 + 2\pi) = \cos\omega t_1} \equiv \cos\left(\omega\left[t_1 + \frac{2\pi}{\omega}\right]\right)$ (due to 2π-periodicity)

Similarly, $\sin(\omega t_1 + 2\pi) = \sin\omega t_1$.

Thus the solution exactly repeats itself after $t_2 - t_1 = 2\pi/\omega$. So $T = 2\pi/\omega$ is called the period of the oscillations.

$$\boxed{\begin{array}{ll} \omega = \text{frequency}, & T = \text{period} \\ \omega = \frac{2\pi}{T} & T = \frac{2\pi}{\omega} \end{array}} \quad (5)$$

5) Existence & uniqueness of solution of a linear DE-2

Thm. 3.1
Let $p(t), q(t), g(t)$ be continuous on $t \in (a,b)$ and let $t_0 \in (a,b)$. Then the sol'n of IVP

$$y'' + p(t)y' + q(t)y = g(t),$$
$$y(t_0) = y_0, \quad y'(t_0) = y_0'$$

exists & is unique for $t \in (a,b)$.

Note: This statement is exactly the same as for a DE-1 (Sec. 2.1), except that here we also have an extra coeff. $q(t)$.

6) Linear DE-2 cannot be solved analytically for general $p(t)$ & $q(t)$.

Note 1 This is in stark contrast to the situation with linear DE-1, which can be solved for any $p(t), g(t)$ (see Lec. 2).

<u>Note 2</u> We'll demonstrate our statement for a <u>homogeneous</u> linear DE-2 ($g(t) = 0$).

Later we'll show that <u>if</u> a homogeneous linear DE-2 <u>can</u> be solved, then so can be its non-homogeneous version, too.

We are going to show that a <u>linear DE-2</u> is equivalent to the <u>Riccati DE-1</u>, which cannot be solved in general (Lec. 7).

Consider

$$y'' + p(t)\,y' + q(t)\,y = 0 \qquad (6)$$

and a <u>change of variables</u>:

$$y = e^{-\underbrace{\int v(t)\,dt}_{V(t)}}, \qquad (7)$$

where $v(t)$ is the new variable.

To substitute (7) into (6), we need y' & y''.

a) $y' \underset{\text{Chain R.}}{=} (-V)' \cdot e^{-V} = -v \cdot e^{-V}$

b) $y'' = (y')' = (-v \cdot e^{-V})' \overset{\text{product R.}}{=} -v' e^{-V} - v \cdot (e^{-V})'$

$= -v' e^{-V} - v \cdot (-v e^{-V}) = -v' e^{-V} + v^2 e^{-V}$.

Substitute these into (6):

$$\underbrace{(-v'\cancel{e^{-V}} + v^2 \cancel{e^{-V}})}_{y''} + p(t)\underbrace{(-v \cdot \cancel{e^{-V}})}_{y'} + q(t) \cdot \underbrace{\cancel{e^{-V}}}_{y} = 0$$

$$-v' + v^2 - p(t)v + q(t) = 0$$

$$v' + p(t)v = v^2 + q(t) \leftarrow \text{Riccati for } v.$$

Since one cannot find v for general p & q, so one cannot find y from (7) for general p & q.

Section 3.1

<u>HW</u>: 3.1. 11 ← graph, concavity

9, 10 ← sol'n of oscillator equ. (Ans. for #10: $C_1 = 1$, $C_2 = -1$.)

13 ← freq. of a bobbing object (see Eq. (3) in book).

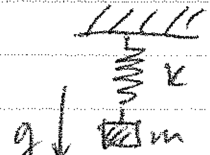
14 ← parameters of osc. model from graph

Ans.: (a) $y_0 = 0$, $T = 2$, (b) $\omega = \pi$, $y_0' = 6\pi$.

3, 7 ← existence/uniqueness interval

<u>WP 1</u>: (a) verify that $x = \sin\omega t$ is a sol. of lin. oscill.
(b) same for $x = c_1\cos\omega t + c_2\sin\omega t$.

<u>WP 2</u>:



$y'' = -\omega^2 y - g$.
Use trick of top p. ⑤, Lec. 2 to find the general sol'n.