Lecture 11. The general solution of a linear homogeneous 2nd-order DE.

We will consider

$$y'' + p(t)y' + q(t)y = 0 \qquad (1)$$

$p(t), q(t)$ are continuous for $t \in (a,b)$.
Then, by Thm. 3.1, $y(t)$ exists & is unique on (a,b).

① Superposition principle

Thm. 3.2 Let $y_1(t), y_2(t)$ be two solutions of (1) and c_1, c_2 be any constants. Then:

$$y_3(t) = c_1 y_1(t) + c_2 y_2(t)$$

is also a sol'n of (1).

Note 1: This is exactly the same as the Superposition Principle (part (a)) for DE-1 (see Lec. 3), and the proof is exactly the same. See also p. 116 in textbook.

Note 2: Later we'll consider the Superposition Principle (part (b)) for non-homogeneous DE-2 and will show that it is also the same as for DE-1.

2/22/16

② Fundamental sets of solutions

2a Motivation and definition

Thm. 3.2 said:

$$\left(\begin{array}{l} y_1, y_2 \text{ are} \\ \text{sol'ns of (1)} \end{array}\right) \Rightarrow \left(\begin{array}{l} y_3 = c_1 y_1 + c_2 y_2 \\ \text{is also a sol'n of (1)} \end{array}\right).$$

Q: Are there two sol'ns $y_1(t), y_2(t)$ of (1) such that any sol'n $y(t)$ of (1) can be written as

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

for some suitable constants c_1, c_2?

Def: Such two solutions y_1 & y_2, if they exist, are called the fundamental set of solutions of (1).

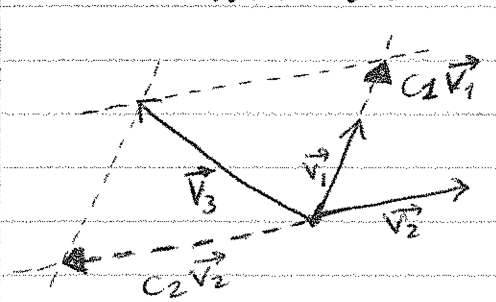
Analogy with vectors in the plane:

Q: Are there two vectors $\vec{v_1}$ & $\vec{v_2}$ such that any vector $\vec{v_3}$ in the same plane can be written as

$$\vec{v_3} = c_1 \vec{v_1} + c_2 \vec{v_2}$$

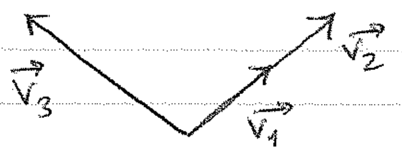
for some c_1, c_2?

A: Any two non-parallel vectors $\vec{v_1}$ & $\vec{v_2}$ will do.



Thus, any two non-parallel $\vec{v_1}$ & $\vec{v_2}$ form a fundamental set of vectors in the plane.

Note: Two parallel vectors will not form a fundamental set:



$\vec{v}_3 \neq c_1\vec{v}_1 + c_2\vec{v}_2$
for any c_1, c_2.

in principle, not analytically

Observations:

1) It is quite likely that one can find a fundamental set of solutions of a DE, just as we've found a fundamental set of vectors in the plane.

2) We can think of solutions of a homogeneous DE as vectors!
Using the definition of a vector space (from Linear Algebra), one can show that they are equivalent. (But we will not show this.)

For a Example of finding c_1, c_2 for solutions of a DE-2 instead of vectors ⟵ MUST SEE EX. 1 in SEC. 3.2

$\boxed{2b}$ What solutions can form a fundamental set? The Wronskian.

Problem: Let $y_1(t), y_2(t)$ be solutions of

$$y'' + p(t)y' + q(t)y = 0. \qquad (1)$$

Find $c_1, c_2 = \text{const.}$ such that

$$y = c_1 y_1 + c_2 y_2 \qquad (2)$$

satisfies the <u>initial condition</u>

$$y(t_0) = y_0, \quad y'(t_0) = y_0'. \qquad (3)$$

(Recall that <u>we know</u> that by Thm. 3.2, this solution exists and is unique.)

<u>Sol'n</u>: Substituting (2) into (3) we find:

$$\begin{aligned} y(t_0) &= \boxed{y_1(t_0)\,c_1 + y_2(t_0)\,c_2 = y_0} \\ y'(t_0) &= \boxed{y_1'(t_0)\,c_1 + y_2'(t_0)\,c_2 = y_0'} \end{aligned} \qquad (4)$$

Recall matrix multiplication:

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1x_1 + a_2x_2 \\ b_1x_1 + b_2x_2 \end{pmatrix}$$

Then Eqs. (4) can be written as <u>one matrix eq.</u>:

$$\underbrace{\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}}_{\substack{\uparrow \\ A}} \underbrace{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}_{\substack{\uparrow \\ \vec{x}}} = \underbrace{\begin{pmatrix} y_0 \\ y_0' \end{pmatrix}}_{\substack{\uparrow \\ \vec{b}}} \qquad (5a)$$

$$A\vec{x} = \vec{b}. \qquad (5b)$$

On one hand, we know from Thm. 3.1 that $y(t)$, which is the solution of the IVP (1) & (3), exists and is unique.

On the other hand, from Linear Algebra we know that the solution $\vec{x}$ of (5b) exists and is unique iff

$$\det(A) \neq 0, \quad \text{i.e.}$$

$$\boxed{W(t_0) \equiv \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \neq 0} \quad (6)$$

"defined as"

New notation: Wronskian of y_1 & y_2.

Thus, so far we have:

$$\left(\begin{matrix} y_1(t), y_2(t) \text{ is a} \\ \text{fundamental set} \end{matrix} \right) \Rightarrow \left(W(t_0) \neq 0 \right). \quad (7a)$$

Conversely, from Lin. Algebra we know that if $\det A \neq 0$ in (5b), then its solution $\vec{x}$ exists for any r.h.s. vector $\vec{b}$.
Then, equivalently, if $W(t_0) \neq 0$ in (5a), then a unique solution $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ always exists for any $\begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$. Since $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ exists, then by (2), the solution of the IVP (1)&(3) can be written as $c_1 y_1 + c_2 y_2$.
Thus:

$$\left(W(t_0) \neq 0 \right) \Rightarrow \left(\begin{matrix} y_1, y_2 \text{ is a} \\ \text{fundamental set} \end{matrix} \right). \quad (7b)$$

To combine (7a) & (7b):

$$\boxed{\begin{pmatrix} y_1(t), y_2(t) \text{ is a} \\ \text{fundamental set} \end{pmatrix} \Longleftrightarrow \left(W(t_0) \neq 0 \right)}. \quad (7c).$$

<u>Note:</u> In a later Lecture we'll show that if (a,b) is the interval of existence of $y_1(t), y_2(t)$, then:

either $\begin{pmatrix} W(t) \neq 0 \\ \text{for } \underline{\text{all}}\ t \in (a,b) \end{pmatrix}$ or $\begin{pmatrix} W(t) = 0 \\ \text{for } \underline{\text{all}}\ t \in (a,b) \end{pmatrix}$

(I.e., $W(t)$ <u>cannot</u> cross zero in the interval of existence of the solution of a DE-2.)

<u>Ex. 1(a)</u> Show that $y_1 = \cos\omega t$, $y_2 = \sin\omega t$ (see Lec. 10, topic 4) form a fundamental set of solutions of the linear oscillator model:

$$y'' + \omega^2 y = 0 \qquad (8)$$

<u>Note:</u> $p(t) = 0$, $q(t) = \omega^2 = \text{const}$ are continuous everywhere, $\Rightarrow$ $y(t)$ exists for <u>all</u> t.

According to the <u>Note above</u>, we just need to compute $W(t_0)$ for some t_0. Instead of name "t_0", just use "t".

<u>Sol'n:</u> $W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos\omega t & \sin\omega t \\ -\omega\sin\omega & \omega\cos\omega t \end{vmatrix}$

$= \omega \cdot \cos^2\omega t - (-\omega \cdot \sin^2\omega t) = \omega\,(\overbrace{\cos^2\omega t + \sin^2\omega t}^{1}) = \omega \neq 0.$

Thus, by (7c), $\{\cos\omega t, \sin\omega t\}$ forms a fundamental set.

__Ex. 1(b)__ Show that $y_1 = \cos\omega t$, $y_2 = A\cos\omega t$ $(A = \text{const})$ is __not__ a fundamental set of (8).

__Sol'n:__ $W(t) = \begin{vmatrix} \cos\omega t & A\cos\omega t \\ -\omega\sin\omega t & A\cdot(-\omega\sin\omega t) \end{vmatrix}$

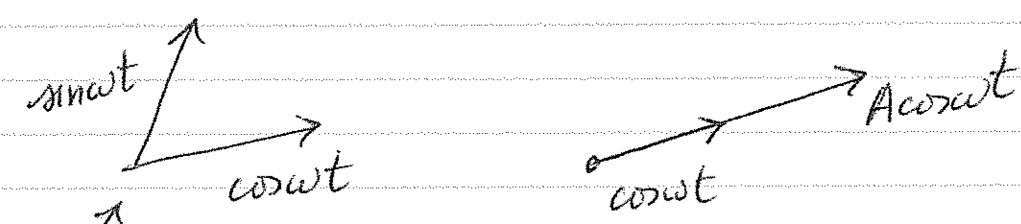
$= A\cdot(-\omega\cdot\cos\omega t\cdot\sin\omega t) - A\cos\omega t\cdot(-\omega\cdot\sin\omega t) = 0.$

Again, by (7c), $\{\cos\omega t, A\cos\omega t\}$ is not a fund. set.

See also __Ex. 3 in Sec. 3.2__

__Analogy with vectors:__

__1(a)__



__fund. set__ __not__ a fund. set.

__HW. Sec. 3.2__
- 1, 3, 6 ($b \to W = \frac{1}{2}e^{t/2}$, $c \to$ Use DSolve), 7, 9, 10 ($b \to W = -\ln 3/t$, $c \to$ Use DSolve), 12, 13 ← Find W and then c_1, c_2.
- 21 (Hint: Use Eqs. (7) of Notes.), 20 ($\alpha \neq \pi n$) ← W & fund. set
- 17, 16 (Hint: See Ex. 1 of Sec. 3.2) ($b \to c_1 = 1/(2\sqrt{2})$, $c_2 = 1/\sqrt{2}$). ← similar to

__WP:__ In Ex. 3 for Sec. 3.2, $W(t)$ crosses zero (at $t=0$). Why does this not contradict the first Note on p. 11-6?