Lecture 12. Homogeneous linear DE-2 with constant coefficients.

We'll consider

$$ay'' + by' + cy = 0, \qquad (1)$$

where $a, b, c = \text{const}$.

1. The idea behind finding solutions of (1).

Recall that if $\lambda = \text{const}$, then

$$(e^{\lambda t})' = \lambda e^{\lambda t}; \quad (e^{\lambda t})'' = \left((e^{\lambda t})'\right)' = \lambda \cdot \lambda e^{\lambda t} = \lambda^2 e^{\lambda t}.$$

Then seek sol'n of (1) as $\boxed{y = e^{\lambda t}}$:

$$a \cdot (e^{\lambda t})'' + b \cdot (e^{\lambda t})' + c \cdot (e^{\lambda t}) = 0$$

$$a \cdot \lambda^2 \cancel{e^{\lambda t}} + b \cdot \lambda \cancel{e^{\lambda t}} + c \cdot \cancel{e^{\lambda t}} = 0$$

Thus, find λ in $y = e^{\lambda t}$, we need to solve

$$\boxed{a\lambda^2 + b\lambda + c = 0} \qquad (2)$$

Characteristic eq. for (1);

$P(\lambda) = a\lambda^2 + b\lambda + c$ ← characteristic polynomial.

Ex. 1 Consider

$$y'' + 8y' + 15y = 0$$

(a) Solve the characteristic eqn. for it.

$$\lambda^2 + 8\lambda + 15 = 0$$

$$\lambda = \frac{-8 \pm \sqrt{64 - 4 \cdot 15}}{2} = \frac{-8 \pm 2}{2} = -4 \pm 1 = \begin{cases} -5 \\ -3 \end{cases}$$

(b) Show that $y_1 = e^{-5t}$ & $y_2 = e^{-3t}$ form a fundamental set (see Lec. 11) of solutions of the DE. Write the general sol'n.

$$W(t) = \begin{vmatrix} e^{-5t} & e^{-3t} \\ -5e^{-5t} & -3e^{-3t} \end{vmatrix} = -3 \cdot e^{-5t} \cdot e^{-3t} - (-5e^{-5t}) \cdot e^{-3t}$$

$$= (5-3) \cdot e^{-5t-3t} = 2e^{-8t} \neq 0 \text{ for all } t,$$

$\Rightarrow$ y_1 & y_2 form a fund. set.

General sol'n: $y = c_1 e^{-5t} + c_2 e^{-3t}$.

② Various cases for roots of the char. eqn.

$$p(\lambda) \equiv a\lambda^2 + b\lambda + c = 0 \qquad (2)$$

$$\Rightarrow \quad \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

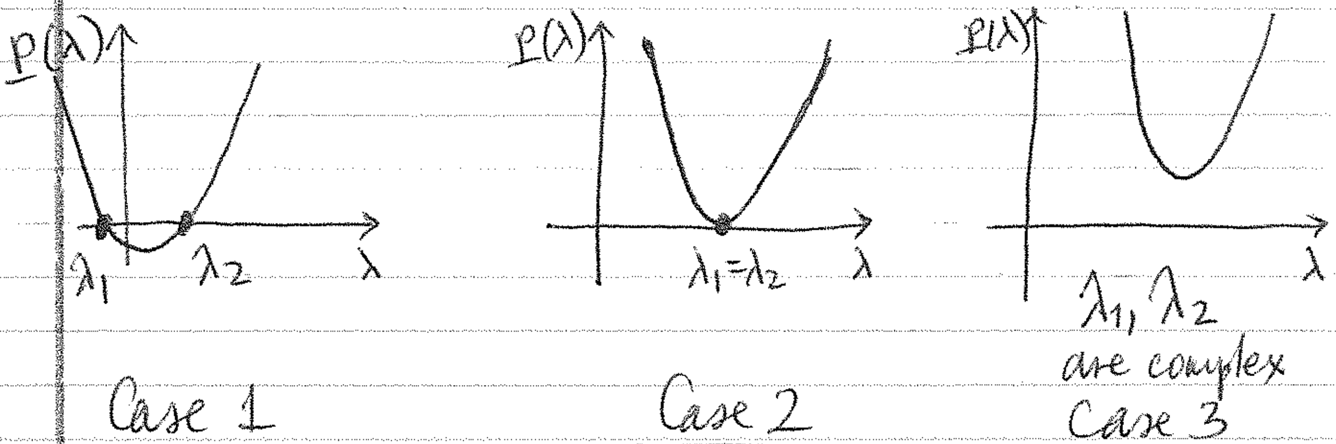
Note: When $a = 1$, the following alternative formula may be convenient:

$$\lambda^2 + p\lambda + q = 0 \qquad (3)$$

$$\Rightarrow \quad \lambda = -\left(\frac{p}{2}\right) \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}.$$

There are 3 cases, depending on the discriminant $(b^2 - 4ac)$ of (2).

<u>Case 1</u> $b^2-4ac>0$; $\Rightarrow P(\lambda)$ has two <u>real</u>, <u>distinct</u> <u>roots</u> λ_1, λ_2.



Then $y_1 = e^{\lambda_1 t}$, $y_2 = e^{\lambda_2 t}$ form a <u>fund. set</u> of sol'ns of (**1**) and $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ is the general sol'n.
See Ex. 1 above & <u>top of p. 124</u> in book.

<u>Case 2</u> <u>$b^2-4ac=0$</u> $\Rightarrow$ <u>$P(\lambda)$</u> has only <u>one</u>, real root $\lambda_1 (=\lambda_2)$.

Then $y_1 = e^{\lambda_1 t}$, and a second sol'n y_2, needed for a fundamental set, is missing. We'll show how to find it in Lec. 13.

<u>Note:</u> $y_2 \neq c\, y_1$: see Ex. 1(b) in Lec. 11 (similar idea, though different functions).

<u>Case 3</u> $b^2-4ac<0$; $\Rightarrow P(\lambda)$ has two <u>complex</u> sol'ns λ_1, λ_2:

$$\lambda_{1,2} = \frac{-b \pm \sqrt{-\underbrace{(4ac-b^2)}_{>0}}}{2a} = \frac{-b \pm i\sqrt{4ac-b^2}}{2a},$$

where $i = \sqrt{-1}$ (or $i^2 = -1$).
A relation of this complex solution to a real-valued sol'n of the IVP will be considered in Lec. 14.

Ex. 2 ($\approx$ Ex. 3 in book); is related to Case 1.

(a) Solve the IVP

$$y'' + y' - 2y = 0, \quad y(0) = y_0, \ y'(0) = y_0'.$$

Sol'n: 1) Characteristic eq:

$$\lambda^2 + \lambda - 2 = 0 \Rightarrow (\lambda+2)(\lambda-1) = 0$$

$\Rightarrow \lambda_1 = -2, \ \lambda_2 = 1.$

2) General sol'n:

$$y = c_1 e^{-2t} + c_2 e^{t}$$

3) IVP:

$$y(0) = c_1 \cdot 1 + c_2 \cdot 1 = y_0$$
$$y'(0) = c_1 \cdot (-2) + c_2 \cdot 1 = y_0'$$

In matrix form: $\underbrace{\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}}_{A} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$

det $A \equiv W = 1 \cdot 1 - (-2) \cdot 1 = 3 \neq 0$, $\Rightarrow$ there is a unique sol'n for any y_0, y_0'.
Finding it (by elimination, e.g.):

$$c_1 = \frac{y_0 - y_0'}{3}, \qquad c_2 = \frac{2y_0 + y_0'}{3}.$$

So: $y = \frac{y_0 - y_0'}{3} \cdot e^{-2t} + \frac{2y_0 + y_0'}{3} \cdot e^{t}$.

(b) For what values of y_0, y_0' can we have $y(t \to \infty) \to 0$? $y(t \to -\infty) \to 0$?

Sol'n: Note that:

$\underline{t \to +\infty}$ $\quad e^{-2t} \to 0, \quad e^{t} \to \infty$

$\underline{t \to -\infty}$ $\quad e^{-2t} \to \infty, \quad e^{t} \to 0$.

Therefore, for $y(t \to +\infty) \to 0$, we must have

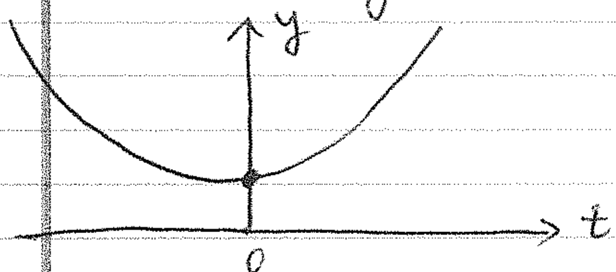
$$C_2 = \frac{2y_0 + y_0'}{3} = 0 \Rightarrow 2y_0 + y_0' = 0.$$

For $y(t \to -\infty) \to 0$, we must have

$$C_1 = \frac{y_0 - y_0'}{3} = 0 \Rightarrow y_0 - y_0' = 0.$$

In all other cases, we'll have

$$y(t \to +\infty) \to \infty \quad \& \quad y(t \to -\infty) \to \infty$$



HW: Sec. 3.3

1, 3, 5, 7, 9, 11, 13 ← general sol, IVP, behavior for $t \to \pm\infty$,

17 ← related to finding λ_2 from $W(t)$; also recover coeff's.

18 ← qualit. behavior; similar to a prob. in Sec. 3.1, but also requires some knowledge of sol'n.

19, 20 ← sol'n of $x''' + bx'' + cx' = 0$ & similar for #20.

EC #6   # 16.