Lecture 14. Complex roots; oscillatory solutions of DE-2

(1) Meaning of the complex exponential.

1a In Case 3 listed in Lecture 12,

$$\lambda_{1,2} = \frac{-b}{2a} \pm i\,\frac{\sqrt{4ac-b^2}}{2a}$$

$$\equiv \alpha \pm i\beta, \qquad (1)$$

if $(4ac - b^2 > 0)$ in the characteristic eq.

$$a\lambda^2 + b\lambda + c = 0 \qquad (2)$$

Then the DE-2:

$$ay'' + by' + cy = 0 \qquad (3)$$

has two solutions

$$y_1 = e^{(\alpha+i\beta)t}, \quad y_2 = e^{(\alpha-i\beta)t}. \qquad (4)$$

Moreover, since $\lambda_1 \neq \lambda_2$, then as shown in Lecture 12 and p. 124 of book,

$\{y_1, y_2\}$ is a FS.

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Ex. 1 Suppose $a=1$, $b=0$, $c=\omega^2$.

Then (2) becomes

$$\lambda^2 + \omega^2 = 0 \quad \Rightarrow$$

$$\lambda^2 = -\omega^2 \Rightarrow \lambda = \pm i\omega . \qquad (5)$$

Then the corresponding DE

$$y'' + \omega^2 y = 0 \qquad (6)$$

has solutions

$$y_1 = e^{i\omega t}, \quad y_2 = e^{-i\omega t}, \qquad (7)$$

and they form a FS.

Q1: From Lec. 10 we know that (6) has a FS $\{\cos \omega t, \sin \omega t\}$. So, how are the complex sol's (7) related to the real sol's $\{\cos\omega t, \sin\omega t\}$?

Q2: In general, what does a complex solution $e^{(\alpha+i\beta)t}$ have to do with real solutions of a DE?

$\boxed{1b}$ Definition of the complex exponential.

In Calculus II, you learned the Maclaurin (or Taylor) series for e^x:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (8)$$

It was proved in Calc. II that this series converges for <u>all real</u> x.

In a higher-level course (Complex Analysis) one proves that it actually converges for <u>all complex</u> x, too, i.e. when

$$x = a + ib.$$

Moreover, in Complex Analysis one also shows that for any x_1 & x_2, one has the <u>familiar rule</u>:

$$\boxed{e^{x_1+x_2} = e^{x_1} \cdot e^{x_2}} \qquad (9)$$

even when x_1, x_2 are complex.

Therefore,

$$e^{\alpha t + i\beta t} = e^{\alpha t} \cdot e^{i\beta t} \qquad (10)$$

Since we know how to compute $e^{\alpha t}$, we just need $e^{i\beta t}$.

$\boxed{1c}$ <u>Euler's formula</u>

We first create a multiplication table for i:

n	0	1	2	3	4	5	6	
i^n	1	i	-1	$-i$	1	i	-1	..... repeats!
			$(i)^2=-1$	$(i)^2 \cdot i$	$i^4 = i^2 \cdot i^2$	$i^4 \cdot i$	$i^5 \cdot i$	

Now use (8):

$$e^{i\beta t} = 1 + \frac{i\beta t}{1!} + \frac{(i\beta t)^2}{2!} + \frac{(i\beta t)^3}{3!} + \frac{(i\beta t)^4}{4!} + \frac{(i\beta t)^5}{5!} + \frac{(i\beta t)^6}{6!} + \dots$$

use Table for i^n

$$\overset{\curvearrowright}{=} 1 + i\cdot\frac{\beta t}{1!} - \frac{(\beta t)^2}{2!} - i\frac{(\beta t)^3}{3!} + \frac{(\beta t)^4}{4!} + i\frac{(\beta t)^5}{5!} - \frac{(\beta t)^6}{6!} + \dots$$

$$= \underbrace{\left[1 - \frac{(\beta t)^2}{2!} + \frac{(\beta t)^4}{4!} - \frac{(\beta t)^6}{6!} + \dots\right]}_{\cos\beta t} + i\cdot\underbrace{\left[\beta t - \frac{(\beta t)^3}{3!} + \frac{(\beta t)^5}{5!} + \dots\right]}_{\sin\beta t}$$

$$= \cos\beta t + i\cdot\sin\beta t.$$

So:

$$\boxed{e^{i\beta t} = \cos\beta t + i\sin\beta t} \qquad (11)$$

Euler's formula

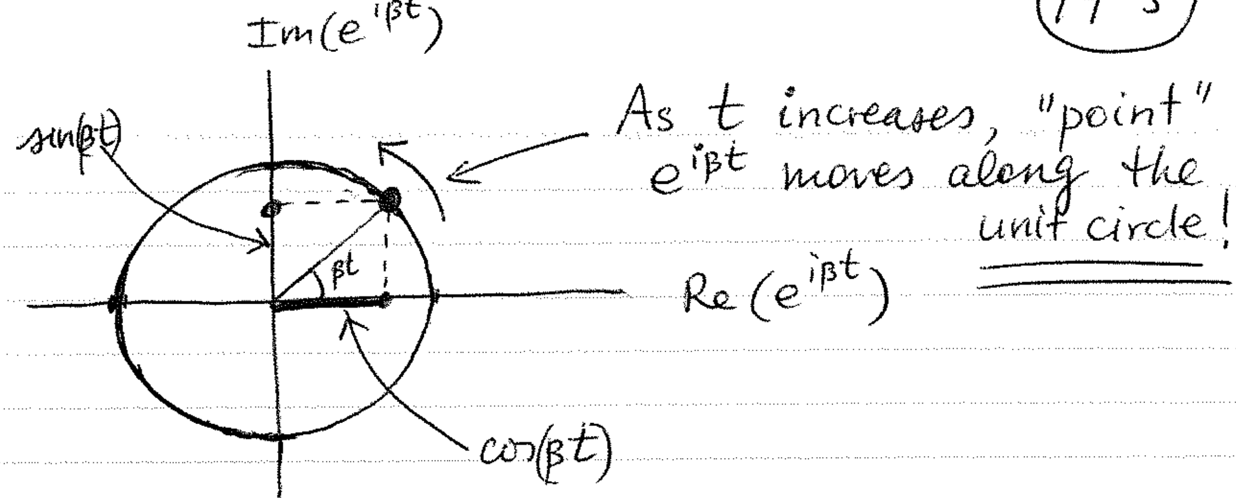
Now: $$e^{-i\beta t} = e^{i(-\beta t)} = \cos(-\beta t) + i\cdot\sin(-\beta t)$$
$$= \cos\beta t - i\cdot\sin\beta t$$

$$\boxed{e^{-i\beta t} = \cos\beta t - i\sin\beta t} \qquad (12)$$

Ex. 2 Unit circle and Euler's formula

$$e^{i\beta t} = \underbrace{\cos\beta t}_{\substack{\text{real part} \\ Re(e^{i\beta t})}} + i\cdot\underbrace{\sin\beta t}_{\substack{\text{imaginary part} \\ Im(e^{i\beta t})}}$$

$Re(a+ib) \equiv a$; $Im(a+ib) \equiv b$.



(2) Constructing a real FS from complex exponentials

[2a] Method 1

$$y_1 = e^{\alpha t + i\beta t} = e^{\alpha t} \cdot e^{i\beta t} = e^{\alpha t}(\cos\beta t + i \sin\beta t) \quad (13a)$$

$$y_2 = e^{\alpha t - i\beta t} = e^{\alpha t} \cdot e^{-i\beta t} = e^{\alpha t}(\cos\beta t - i \cdot \sin\beta t) \quad (13b)$$

Since $\{y_1, y_2\}$ is a FS, then

$z_1 = \frac{y_1 + y_2}{2}$, $z_2 = \frac{y_1 - y_2}{2i}$ are also sol'ns of DE (3) (because any $C_1 y_1 + C_2 y_2$ is a solution).

Thus: $$\boxed{z_1 = e^{\alpha t} \cdot \cos\beta t \ ; \quad z_2 = e^{\alpha t} \cdot \sin\beta t} \quad (14)$$

is another pair of solutions of (3).

Q: Do they form a FS?

A: Yes. See top of p. 136.

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$\boxed{2b}$ Method 2

<u>Thm. 3.3</u> (simplified version for DE-2 with <u>constant</u> coefficients; see book for general case).

Consider DE (3):

$$ay'' + by' + cy = 0 \qquad (3)$$

with a, b, c = <u>real</u> constants.
Let

$$y = e^{\alpha t + i\beta t} = e^{\alpha t}(\cos\beta t + i \cdot \sin\beta t)$$
$$= \underbrace{e^{\alpha t}\cos\beta t}_{u(t)} + i \cdot \underbrace{e^{\alpha t} \cdot \sin\beta t}_{v(t)}$$

be a sol'n of (3). Then $u(t)$ and $v(t)$ <u>individually</u> are also sol'ns of (3).

<u>Proof</u>: $ay'' + by' + cy = 0$

$$a(u + iv)'' + b(u + iv)' + c \cdot (u + iv) = 0$$

$$\underbrace{(au'' + bu' + cu)}_{\text{real}} + \underbrace{i(av'' + bv' + cv)}_{\text{purely imaginary}} = 0$$

This can hold only when "real $= 0$" <u>and</u> "imaginary $= 0$"

$\Rightarrow$ $au'' + bu' + cu = 0$ and

$av'' + bv' + cv = 0$.

Thus, we've shown that

$$u = e^{\alpha t}\cdot \cos\beta t, \quad v = e^{\alpha t}\cdot \sin\beta t$$

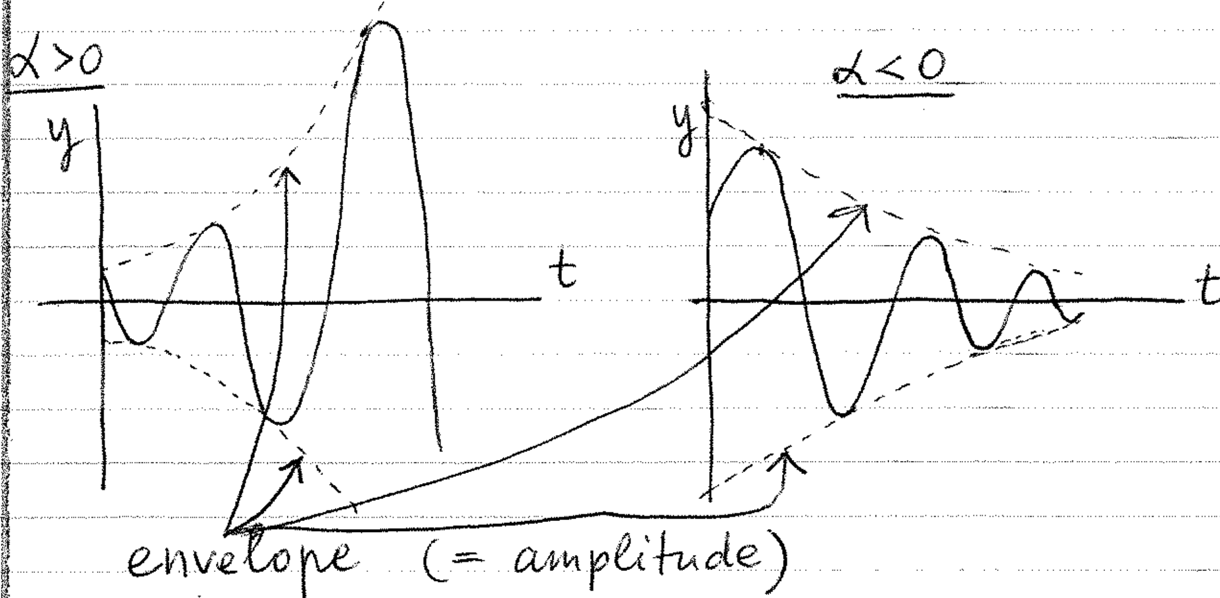
are sol'ns of (3).

2c The general solution

$$\boxed{\begin{aligned} y &= c_1 \overset{u}{z_1} + c_2 \overset{v}{z_2} \\ &= c_1 e^{\alpha t} \cos\beta t + c_2 e^{\alpha t}\cdot \sin\beta t \\ &\equiv A e^{\alpha t} \cos\beta t + B e^{\alpha t} \sin\beta t. \end{aligned}} \quad (15)$$

See Ex. 2 in book for using this general sol'n to solve an IVP.

Note: Let us plot a representative solution (for some A & B)



③ Amplitude and phase of solution

Trig. identity:

$$\boxed{\cos(a-b) = \cos a \cdot \cos b + \sin a \cdot \sin b}$$

note the sign!

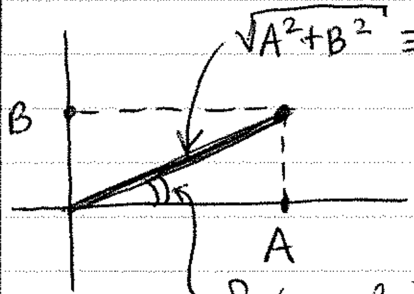
Sanity check: $a = b$

$$\underbrace{\cos(a-a)}_{1} = \underbrace{\cos a \cdot \cos a}_{\cos^2 a} + \underbrace{\sin a \cdot \sin a}_{\sin^2 a} \quad \checkmark$$

Now let us "walk backwards" from r.h.s. to l.h.s.

$$A\cos\beta t + B\sin\beta t = \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\cos\beta t + \frac{B}{\sqrt{A^2+B^2}}\sin\beta t\right)$$

$\sqrt{A^2+B^2} \equiv R$ (radius in polar coord's)



δ (angle)

$$\boxed{\tan\delta = \frac{B}{A}}$$

$$\cos\delta = \frac{A}{\sqrt{A^2+B^2}}$$

$$\sin\delta = \frac{B}{\sqrt{A^2+B^2}}$$

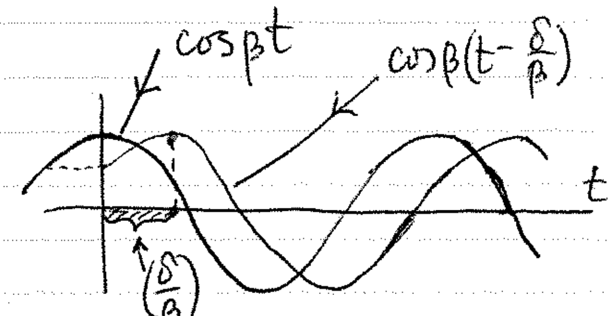
Then

$$\frac{A}{\sqrt{A^2+B^2}}\cos\beta t + \frac{B}{\sqrt{A^2+B^2}}\sin\beta t = \cos\delta\cos\beta t + \sin\delta\sin\beta t$$

$$= \cos(\beta t - \delta).$$

Meaning of "$-\delta$"

$$\cos(\beta t - \delta) = \cos\left(\beta\left[t - \frac{\delta}{\beta}\right]\right)$$



So $\left(\frac{\delta}{\beta}\right)$ represents the "phase shift" of the cosine function.

> Note: the book incorrectly refers to δ as the phase!

Now look at the general sol'n of DE (3):

$$y = e^{\alpha t}(A\cos\beta t + B\sin\beta t)$$

$$= e^{\alpha t}\cdot\sqrt{A^2+B^2}\cdot\left(\frac{A}{\sqrt{A^2+B^2}}\cos\beta t + \frac{B}{\sqrt{A^2+B^2}}\sin\beta t\right)$$

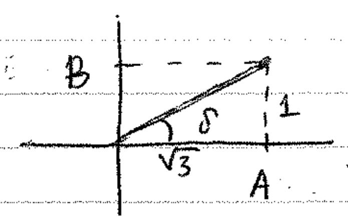
$$= e^{\alpha t}\cdot R\cos(\beta t - \delta) = e^{\alpha t}\cdot R\cdot\cos\beta\left(t - \frac{\delta}{\beta}\right).$$

Ex. 3(a) Rewrite

$$y = \sqrt{3}\cdot\cos\frac{t}{2} + \sin\frac{t}{2}$$

in the form $R\cos(\beta t - \delta)$ and sketch it.

Sol'n:

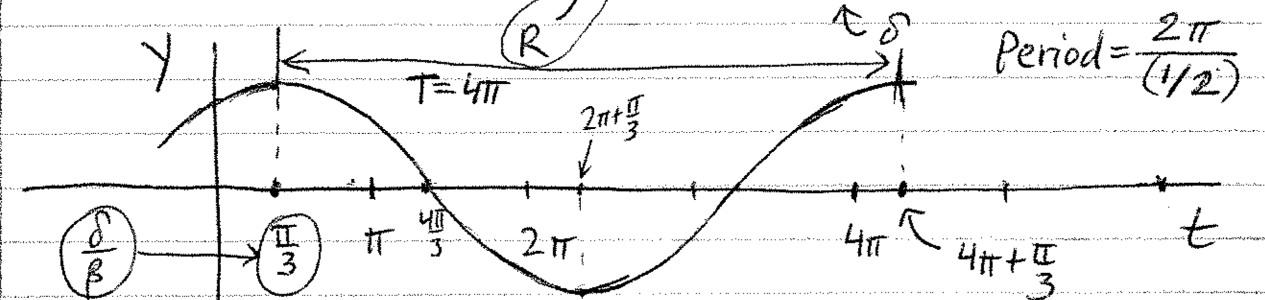


$$\delta = \arctan\frac{B}{A} = \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}.$$

$$R = \sqrt{A^2+B^2} = \sqrt{3+1} = 2.$$

$$y = \underset{R}{2}\cos\left(\frac{t}{2} - \underset{\delta}{\frac{\pi}{6}}\right) = 2\cos\left[\underset{\beta}{\boxed{\tfrac{1}{2}}}\left(t - \frac{\pi}{3}\right)\right].$$

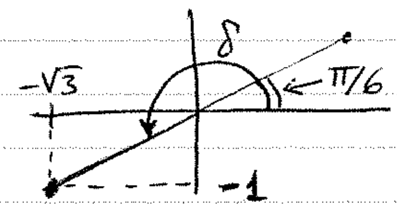
$$\text{Period} = \frac{2\pi}{(1/2)}$$



Ex. 3(b) Same for

$$y = -\sqrt{3}\cos\frac{t}{2} - \sin\frac{t}{2}.$$

Sol'n:

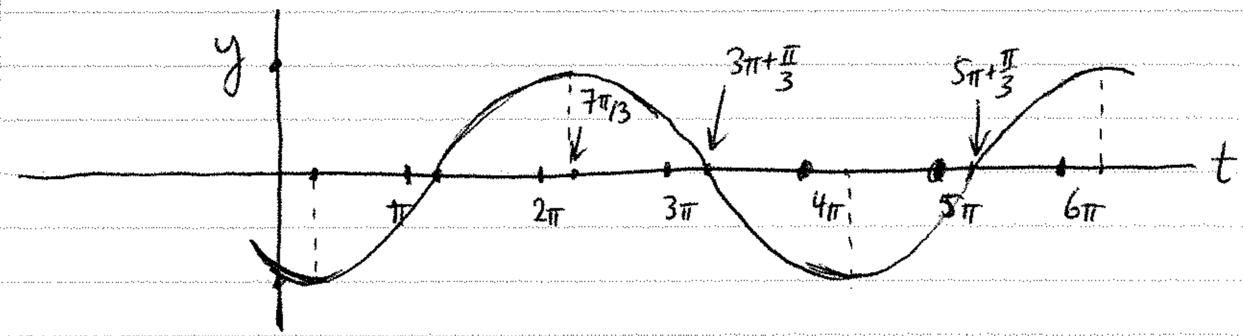


$$\delta = \arctan\left(\frac{-1}{-\sqrt{3}}\right) \underbrace{+\,\pi}_{\text{need to add when } A<0}$$

$$= \frac{\pi}{6} + \pi,$$

$$y = 2\cos\left(\frac{1}{2}t - \left(\frac{\pi}{6}+\pi\right)\right) = 2\cos\left[\frac{1}{2}\left(t - \frac{7\pi}{3}\right)\right]$$

as in 3(a)



Not surprisingly, this looks like the graph in Ex. 3(a), but flipped about the x-axis.

HW. Sec. 3.5. 1, 2 — polar to Cartesian; use ComplexExpand

3, 7, 9, 11 ← find $\lambda = \alpha + i\beta$, gen. sol, sol. of IVP

13, 15, 19, 21 ← from given sol'n find a, b, y_0, y_0'. (19, 21 — expand the cos first)

23, 24, 25, 26 ← rewrite $A\cos t + B\sin$ as $R\cos(\ldots - \delta)$.

(for 24, 26 — use TrigFactor)

27, 28, 29 — rewrite as $R\cos(\beta t - \delta)$ from graph (#28: $\cos(\frac{3}{2}t - \pi/8)$) amplitude

Hint: 1) find period T; 2) $\beta = \frac{2\pi}{T}$ (Lec. 10); 3) find δ/β; 4) $R\cos(\beta t - \delta)$.

31 ← find λ_1, λ_2 and use f-la for W from top of p. 124.

33 ← roots of complex DE; Re y is not a sol'n. || 32 ← buoyancy w/drag.