Lecture 16. General sol'n of linear nonhomogeneous DE.

Here we will obtain the counterpart of the Superposition Principle (part (b)) for linear 2nd-order DEs.

$$y'' + p(t)y' + q(t)y = g(t) \qquad (1)$$

1 The General sol'n of a NH DE-2.

Q: Suppose $u(t)$ & $v(t)$ are two sol'ns of (1), How different can they be? I.e., what can one say about $(u(t)-v(t))$?

A:
$$u(t) - v(t) = y_c(t),$$
where $y_c(t)$ is a sol'n of the homogeneous version of (1).

Proof: Since u, v solve (1) $\Rightarrow$

$$\begin{array}{rl} & u'' + pu' + qu = g \\ - & v'' + pv' + qv = g \\ \hline & (u-v)'' + p(u-v)' + q(u-v) = \cancel{g-g} \to 0; \end{array}$$

$\Rightarrow$ $(u-v)$ satisfies the homogeneous version of DE. ✓

Notations:

- A solution of the NH DE (1) is called particular sol'n of NH DE: y_p.

- The general sol'n of the homogeneous version of (1) is called the <u>complementary</u> sol'n : y_c .

Then the above result implies:

$$\begin{pmatrix}\text{The }\underline{\text{general}}\\ \text{sol'n of a NH DE}\end{pmatrix} = \begin{pmatrix}\text{A particular}\\ \text{sol'n of}\\ \text{NH DE}\end{pmatrix} + \begin{pmatrix}\text{The general}\\ \text{sol'n of the}\\ \underline{\text{homogeneous}}\\ \text{DE}\end{pmatrix}$$

$$\boxed{y(t) = y_p + \underbrace{(c_1 y_1(t) + c_2 y_2(t))}_{y_c}} \quad (2)$$

Here $\{y_1, y_2\}$ is a FS of sol'ns of the homogeneous DE.

<u>Note</u>: In Lecture 3 we had:
If y_h is a solution of the hom. DE-1; and if y_{nh} is a sol'n of the non-hom. DE-1, then $y = c y_h + y_{nh}$
is a sol'n of the non-hom. DE-1.
So, Eq. (2) is just a generalization of that result for a DE-2.

<u>See Ex. 1 in texbook.</u>

(2) <u>Superposition Principle for particular sol'ns.</u>

Similarly to Note 1 on p. 3-6 (Lec. 3), if y_{p_1} and y_{p_2} are two particular sol'ns of the same NH DE, then $(y_{p_1} + y_{p_2})$ is <u>not</u> a sol'n of that DE. (Proof: WP2 for HW#3.)

However, the following useful modification of the superposition principle for NH DE does hold:

<u>Thm. 3.4</u> Let u_1 be a sol'n of

$$y'' + p(t)y' + q(t)y = g_1(t);$$

let u_2 be a sol'n of

$$y'' + p(t)y' + q(t)y = g_2(t);$$

let p, q, g_1, g_2 be continuous on $t \in (a,b)$; and let A_1, A_2 be any <u>constants</u>. Then

$$y_p = A_1 u_1 + A_2 u_2 \qquad (3)$$

is a particular sol'n of

$$y'' + p(t)y' + q(t)y = A_1 g_1(t) + A_2 g_2(t). \qquad (4)$$

<u>Proof</u>: Similar to that on p. 16-1; see a HW problem.

See Ex. 2 in textbook.

<u>Note 1</u>: The (obvious) significance of Thm. 3.4 is that it allows one to find a particular sol'n of a NH DE whose r.h.s. consists of several terms.
You simply find (see Secs. 3.8 & 3.9) particular sol'ns corresponding to <u>individual</u> terms and then "put them together" as "building blocks".

Note 2: Recall (similarly to a statement in Lecture 3) that Superposition Principles are valid only for linear DEs.

Neither of Superposition Principles of this Lecture is valid for a nonlinear DE.

HW: Sec. 3.7

1, 3, 7, 11 ← verify y_p, find $y_{general}$, solve IVP.

13 ← prove Thm. 3.4.

14, 15 ← illustrate #13.

17 ← find $g(t)$ if given y_p

23, 25 ← find $g(t)$ and const. coeff. of DE if $y_{gen.}$ is given.