Lecture 17 Particular sol'n of a non-homogeneous DE – 2: the method of undetermined coefficients.

In Lec. 16 we learned that to find the general sol'n of a DE, we need its particular solution. In this Lecture we'll learn one method of finding a part. sol'n. This method works in some "simple" cases. In Lec. 18 we'll learn a more general method that works in all cases.

In all Examples in this Lecture the DE is:

$$ay'' + by' + cy = g(t), \qquad (1)$$

$a, b, c =$ const (in fact, $a = 1$).

Also, denote:

$y_h(t) =$ solution of the homogeneous version of (1).

Ex. 1 $g(t) = e^{\alpha t} \neq y_h(t)$

$$y'' - y' - 2y = G \cdot e^{\alpha t}, \quad G = \text{const}.$$

Find y_p, given that $e^{\alpha t} \neq y_h(t)$.

Sol'n: 1) Let's first see what $y_h(t)$ is.

$$\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda_{1,2} = -1, 2 \qquad (2)$$

Thus, $\alpha \neq -1$ or 2.

2) Seek $y_p = Ae^{\alpha t}$

$y_p' = \alpha A e^{\alpha t}$, $y_p'' = \alpha^2 A e^{\alpha t}$

Substitute into (1):

$\alpha^2 A e^{\alpha t} - \alpha A e^{\alpha t} - 2Ae^{\alpha t} = Ge^{\alpha t}$

$A\cancel{e^{\alpha t}}(\alpha^2 - \alpha - 2) = G\cancel{e^{\alpha t}}$

$A = \dfrac{G}{\alpha^2 - \alpha - 2}$.

__Note:__ Since $\alpha \neq -1$ or $2 \Rightarrow$
$\alpha^2 - \alpha - 2 \neq 0 \Rightarrow$
can find A.
See Ex. 1 in ~~book~~ ~~for~~ ~~##~~.

__Ex. 2__ $g(t) = t^n$ $(\neq y_h(t))$.

Find y_p for

$$y'' - y' - 2y = t^2.$$

__Note 1__: t^n is not a sol'n of a homogen. DE with const. coefficients for any n.

__Sol'n:__ Seek $\boxed{y_p = A_2 t^2 + A_1 t + A_0}$ (3)

Q: Why not just At^2?

A: a) An *explicit* reason why this won't work will be clear soon.
b) A general (philosophical) reason is:

$g(t) = t^2$ is a special case of a 2nd-degree polynomial. Then y_p is to be sought in the form of the most general 2nd-degree poly.

$$y_p' = A_2 \cdot 2t + A_1 \cdot 1$$
$$y_p'' = A_2 \cdot 2$$

Substitute into DE:

$$(2A_2) - (2A_2 t + A_1) - 2(A_2 t^2 + A_1 t + A_0) = 1 \cdot t^2$$

Collect powers of t:

@ t^2: $\quad -2A_2 = 1 \quad$ (I)

@ t^1: $\quad -2A_2 - 2A_1 = 0 \quad$ (II)

@ t^0: $\quad 2A_2 - A_1 - 2A_0 = 0 \quad$ (III)

Note 2: Going back to the Question above, we see that $y_p = A_2 t^2$ (i.e. w/o A_1, A_0) would not have been able to satisfy all of the 3 eqs. (I), (II), (III). Hence we needed 3 coefficients = # of coefficients in the most general 2nd-degree polynomial.

Continuing with the solution:

Find A_2 from (I), then A_1 from (II), then A_0 from (III):

(I) $\Rightarrow$ $A_2 = -1/2$

(II) $\Rightarrow$ $A_1 = -A_2 = 1/2$

(III) $\Rightarrow$ $A_0 = \frac{1}{2}(-1 - 1/2) = -3/4$

Thus $y_p = -\frac{1}{2}t^2 + \frac{1}{2}t - \frac{3}{4}$.

Note 3 : The general sol'n is

$y(t) = c_1 e^{-t} + c_2 e^{2t} + y_p$.

Ex. 3 $g(t) = \{\sin\beta t \text{ or } \cos\beta t\} \neq y_h(t)$.

$$y'' - y' - 2y = G\sin(\beta t); \quad (G = \text{const})$$

find ~~a~~ y_p.

Sol'n: Seek

$$\boxed{y_p = A\sin\beta t + B\cos\beta t} \qquad (4)$$

Q: Why not just $A\sin\beta t$?

A: a) An explicit reason will be clear soon.

b) The general (philosophical) reason:

- In Lec. 14 we saw that

$$e^{i\beta t} = \cos\beta t + i\cdot\sin\beta t$$

Thus, cos & sin are <u>not</u> <u>unrelated</u> functions, but are two "parts" of one exponential function.

See ##38,39 in Sec. 3.8 for more details

- In Lec. 14 we also explored the identity

$$\cos\left[\beta\left(t-\frac{\delta}{\beta}\right)\right] \equiv \cos(\beta t-\delta) = \cos\beta t\cdot\cos\delta + \sin\beta t\cdot\sin\delta,$$

$\Rightarrow$ cos & sin are two "parts" of a "shifted cos", $\cos\beta(t-\delta/\beta)$.

Thus, by taking a linear combination (4) of cos & sin, we are taking <u>the most general</u> form of the appropriate trig. function.

Continuing with the solution:

$$(4) \Rightarrow y_p' = \beta A\cdot\cos\beta t - \beta B\sin\beta t$$

$$y_p'' = -\beta^2 A\sin\beta t - \beta^2 B\cos\beta t.$$

Substitute into DE:

$$(-\beta^2 \underline{A\sin\beta t} - \beta^2 B\cos\beta t) - (\beta A\cos\beta t - \underline{\beta B\sin\beta t})$$

$$-2(\underline{A\sin\beta t} + B\cos\beta t) = G\cdot\underline{\sin\beta t}$$

Collect terms at $\sin\beta t$ & $\cos\beta t$ separately:

@ sin: $A\cdot[-\beta^2-2] + B\cdot[\beta] = G \quad | \cdot\beta$

@ cos: $A\cdot[-\beta] + B\cdot[-\beta^2-2] = 0. \quad | \cdot(-\beta^2-2)$ +

Multiply as shown above and add:

$A\cdot 0$ (by design) $+ B\cdot[\beta^2+(-\beta^2-2)^2] = G\cdot\beta$

$\Rightarrow \quad B = G\beta / [\beta^2 + (-\beta^2-2)^2]$

similarly:

$A = G\cdot(-\beta^2-2)/[\beta^2+(-\beta^2-2)^2].$

Note 1: It is clear that with just
$y_p = A\sin\beta t$
we would not have been able to satisfy two eqs. to match the sin & cos terms.

Note 2: Since $\beta^2+(-\beta^2-2)^2 \neq 0$
(in fact, it is >0),
we can solve for A & B no matter what G and β are.
See Ex. 3 in book for numbers.

Ex. 4 $\quad g(t) = e^{\alpha t} = y_h(t).$

Find y_p for

$$y'' - y' - 2y = Ge^{2t}.$$

Note 1: e^{2t} is one of the hom. sol'ns, since $\lambda_{1,2} = -1, 2$.

So, if we try to substitute $y_p = Ae^{2t}$, as in Ex. 1, we'll get:

$$A\underbrace{(2^2-2-2)}_{0}e^{2t} = Ge^{2t}$$

$$A \cdot 0 = G \Rightarrow \text{no solutions.}$$

Sol'n: Trick: seek

$$\boxed{y_p = A \cdot t \cdot e^{2t}} \qquad (5)$$

$$y_p' = 2A\cdot t\cdot e^{2t} + A\cdot e^{2t}$$

$$y_p'' = (2^2 A t e^{2t} + 2A\cdot e^{2t}) + 2Ae^{2t}.$$

Substitute into DE:

$$A\left[(2^2\cdot te^{2t} + 2\cdot 2e^{2t}) - (2te^{2t} + e^{2t}) - 2te^{2t}\right]$$

$$= Ge^{2t}$$

$$A\cdot\left[\underbrace{(2^2-2-2)}_{=0}\cdot te^{2t} + (4-1)e^{2t}\right] = Ge^{2t}$$

$$A\cdot 3\cdot e^{2t} = Ge^{2t} \Rightarrow A = G/3.$$

Thus, $y_p = \frac{G}{3} te^{2t}$.

Note 2: The above trick is similar to that used in Lec. 13 and earlier in Lec. 2:

$$\boxed{Y_{new} = Y_{old}\cdot u(t)}. \qquad (6)$$

In fact, substituting (6) into DE, one can verify that in this case, $u(t) \equiv t$ indeed.

Note 3 : When the DE has repeated λ (as in Lec. 13), a more general substitution than (5) will be needed. See Ex. 5 in book.
Note that substitution (6) will still give an appropriate form of $u(t)$, but with more work.

Note 4 : On your own:

- MUST READ Ex. 6 (a, c, d) / book;
- Use Table on p. 163 to find the appropriate form of y_p (see also Ex. 6 in these Notes below).

!

Ex. 5 $g(t) = \{\sin\beta t \text{ or } \cos\beta t\} = y_h(t)$

Find y_p for

$$y'' + \omega^2 y = G \sin\omega t.$$

(Here, $g(t) = \sin\omega t$ is a sol'n of the homogeneous DE; see Lec. 10 & 14.)

Sol'n: Combining the ideas of Ex. 3 and Ex. 4, we seek:

$$\boxed{y_p = t\cdot(A\sin\omega t + B\cos\omega t)} \quad (7)$$

$y_p' = t\omega(A\cos\omega t - B\sin\omega t) + (A\sin\omega t + B\cos\omega t)$

$y_p'' = -t\omega^2(A\sin\omega t + B\cos\omega t) + 2\omega(A\cos\omega t - B\sin\omega t)$.

Substitute into DE:

$\left[-t\cdot\omega^2(A\sin\omega t + B\cos\omega t) + 2\omega(A\cos\omega t - B\sin\omega t)\right]$

$+ \omega^2\left[t(A\sin\omega t + B\cos\omega t)\right] = G\cdot\sin\omega t$

The underlined terms exactly cancel, ⇒

$2\omega(A\cos\omega t - B\sin\omega t) = G\cdot\sin\omega t.$

@ $\sin\omega t$: $-2\omega B = G \Rightarrow B = \frac{-G}{2\omega}$

@ $\cos\omega t$: $2\omega\cdot A = 0 \Rightarrow A = 0.$

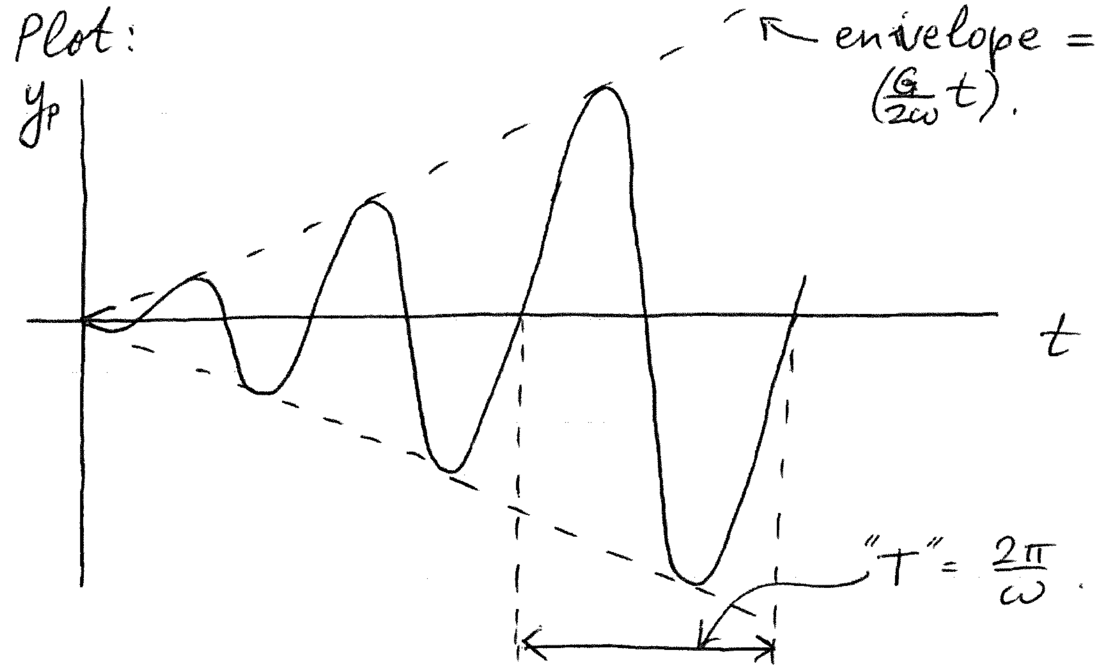
Thus $\boxed{y_p = -\frac{G}{2\omega}\cdot t\cdot\cos\omega t.}$

Important Note: This Example describes an important physical phenomenon known as the resonance. It describes a solution whose amplitude grows in time, even though the r.h.s. $g(t) = G\sin\omega t$ does not grow.

This happens only when:

- (y') - term is absent; and
- (ω on rhs) equals (ω on l.h.s.)

Plot:



HW : General note: see Table on p. 163 for the form of y_p in each case.

1, 3, 4 ← find y_p; $g \neq y_h$
5, 11, 13 ← find y_p, $g = y_h$ (exponential)
17, 19 ← general form of y_p
29, 30 ← given y_p & g, find α, β in $y'' + \alpha y' + \beta y = g$.
31 ← given asymptotic behavior for $g \neq y_h$, find IC.

Find y_p:

WP1 $y'' + \omega^2 y = \cos\omega t$
WP2 $y'' + 2\alpha y' + \omega^2 y = \cos\omega t$
WP3 $y'' + \omega^2 y = \cos(0.9\omega t)$

Plot all three sol'n together (use Mathematica) for:
$\omega = 1$, $\alpha = 0.1$; $t \in [0, 10\pi]$.