Lecture 18 Particular sol'n of a nonhomogeneous DE-2: the method of variation of parameters.

1 Brief review of DE-1 case

Solve $\quad y' + p(t)y = g(t) \qquad (1)$

given y_h: $y_h' + p(t)\,y_h = 0. \qquad (2)$

Solution: seek $\quad y = y_h \cdot u(t) \quad \Rightarrow$

$$(y_h \cdot u' + \underbrace{y_h' u) + p \cdot y_h \cdot u}_{=0 \text{ by } (2)} = g$$

$$\Rightarrow \quad y_h \cdot u' = g \quad \Rightarrow \quad u' = g/y_h$$

$$\Rightarrow \text{solve for } u \quad \Rightarrow \quad y = y_h \cdot u. \quad \checkmark.$$

2 Formula for DE-2 case

Solve $\quad y'' + p(t)y' + q(t)y = g(t) \qquad (3)$

given that $\{y_1, y_2\}$ form a FS of the homogeneous DE-2

$$y'' + py' + qy = 0. \qquad (4)$$

Use a similar "trick": seek $\qquad$ Trick #1

$$\boxed{y_p(t) = y_1 \cdot u_1(t) + y_2 \cdot u_2(t)}, \qquad (5)$$

where $u_1(t)$, $u_2(t)$ are TBD.

$$y_p' = (\underline{y_1' u_1} + y_1 u_1') + (\underline{y_2' u_2} + y_2 u_2')$$

rearrange $$\stackrel{*}{=} [\underline{y_1' u_1} + \underline{y_2' u_2}] + [y_1 u_1' + y_2 u_2']$$

new

Before we compute y_p'', use Trick #2:

Require:
$$\boxed{y_1 u_1' + y_2 u_2' = 0} \quad (6)$$

Continue while using (6):

$$y_p' = [y_1' u_1 + y_2' u_2] \quad \Rightarrow$$

$$y_p'' = y_1'' u_1 + y_1' u_1' + y_2'' u_2 + y_2' u_2'.$$

Substitute into (3):

$$(\underline{y_1'' u_1} + y_1' u_1' + \underline{\underline{y_2'' u_2}} + y_2' u_2') + p \cdot (\underline{y_1' u_1} + \underline{\underline{y_2' u_2}})$$
$$+ q \cdot (\underline{y_1 u_1} + \underline{\underline{y_2 u_2}}) = g(t)$$

Rearrange:

$$\underbrace{\cancel{(y_1'' + p y_1' + q y_1)}}_{0} \cdot u_1 + \underbrace{\cancel{(y_2'' + p y_2' + q y_2)}}_{0} \cdot u_2 \qquad 0 \xleftarrow{} \text{by (4)} \xrightarrow{} 0$$
$$+ [y_1' u_1' + y_2' u_2'] = g$$

$$\Rightarrow \quad \boxed{y_1' u_1' + y_2' u_2' = g} \quad (7)$$

Note 1: Had we not required (6), then y_p'' would have had terms like $y_1 u_1''$ & $y_2 u_2''$, and (7) would not have had its simple form.

However, a systematic explanation for trick #2 will be given only in Chap. 4.

Continuing with the solution:

$$\begin{aligned} (6):&\quad y_1 u_1' + y_2 \cdot u_2' = 0 \qquad &\Big|\ \cdot y_2' \\ (7):&\quad y_1' u_1' + y_2' u_2' = g. \qquad &\Big|\ \cdot y_2 \end{aligned}$$

This is a linear system for $\{u_1', u_2'\}$:

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \cdot \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix} \qquad (8)$$

Its solution exists for any g because

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W(t) \neq 0,$$ because $\{y_1, y_2\}$ is a FS of sol'ns.

Solve (6) & (7).

E.g., $(6)\cdot y_2' - (7)\cdot y_2$ (see above) yields:

$$(y_1 y_2' - y_1' y_2) u_1' + \underbrace{\cancel{(y_2 y_2' - y_2' y_2)}}_{0} u_2' = 0 - g \cdot y_2$$

$$\Rightarrow \quad \boxed{u_1' = \frac{-g \cdot y_2}{W(t)}} \qquad (9a)$$

Similarly:

$$\boxed{u_2' = \frac{g \cdot y_1}{W(t)}} \quad . \qquad (9b)$$

Integrating:

$$u_1(t) = -\int^t \frac{g(s)\, y_2(s)}{W(s)}\, ds \qquad (10a)$$

$$u_2(t) = \int^t \frac{g(s)\, y_1(s)}{W(s)}\, ds \qquad (10b)$$

Combine (10) & (5):

$$\boxed{y_p(t) = -y_1(t)\int^t \frac{g(s)\, y_2(s)}{W(s)}\, ds + y_2(t)\int^t \frac{g(s)\, y_1(s)}{W(s)}\, ds} \qquad (11)$$

!

Note 2: The upper limit "t" above indicates that after the integration, we should set the variable $s = t$.

The lower limit is not important because it will only shift the integral by a constant. Then sol'n (11) will shift by:

$$a_1 \cdot y_1 + a_2 y_2 \equiv y_c(t) \leftarrow \text{sol'n of homogeneous DE},$$

and changing y_p by some $y_c(t)$ does not affect the general solution

$$y(t) = y_p(t) + C_1 y_1 + C_2 y_2 . \qquad (12)$$

3 Examples

Ex. 1 $y'' - y' - 2y = Ge^{2t}$

Find y_p.

Note 1: We solved this in Lec. 17 by a substitution

$$y_p = A \cdot t e^{2t} \qquad (13)$$

(because $y_h = e^{2t}$ is a sol'n of the homogeneous problem).
However, this substitution was postulated but not derived.

Now we will derive (13).

Sol'n:

1) Ex. 4 in Lec. 17 $\Rightarrow$

$$y_1 = e^{-t}, \quad y_2 = e^{2t}; \quad W(t) = 3e^{t}$$ (p. 124 of book)

We need $u_1(t)$, $u_2(t)$.
Let's compute $u_2(t)$ and show that term

$$y_2 \cdot u_2$$

agrees with (13)

Note 2: In HW 18 you may explore what happens to term $y_1 u_1$.

2) From formula (10b):

$$u_2(t) = \int^t \frac{g(s)\, y_1(s)}{W(s)}\, ds = \int^t \frac{G e^{2s} \cdot e^{-s}}{3 e^{s}}\, ds =$$

$$= \int_{t_0}^{t} \frac{G}{3}\,ds = \frac{G}{3} s \Big|_{t_0}^{t} = \frac{G}{3}t - \underbrace{\frac{G}{3}t_0}_{=\text{const}}.$$

(t_0: some const)

Thus,

$$y_2 \cdot u_2 = y_2 \cdot \underbrace{\left(\frac{G}{3}t - \frac{G}{3}t_0\right)}_{u_2} =$$

$$= \underbrace{\underbrace{\frac{G}{3}}\, t e^{2t}}_{\text{same as in Ex. 4 of Lec. 17}} + \underbrace{\underbrace{\left(-\frac{G}{3}t_0\right)}_{\text{some const}} \cdot e^{2t}}_{\text{part of the homogeneous sol'n.}}$$

Skip to Ex. 3

Ex. 2 Find the general sol'n of

$$t^2 y'' - t y' + y = t$$

Sol'n:

1) Always put into form

coefficient $= 1$ → $y'' + p y' + q y = g,$

because previous formulas were derived for this form (i.e. when the coefficient of y'' is 1).

So:

$$y'' - \frac{1}{t}y' + \frac{1}{t^2}y = \frac{1}{t}$$

$\Rightarrow g(t) = 1/t.$

2) In #15 of HW13 you showed that

$$y_1 = t, \quad y_2 = t\ln t$$

form a fundamental set of sol'ns for the homogeneous version of this DE.

$$W(t) = \begin{vmatrix} t & t\ln t \\ 1 & (\ln t)+1 \end{vmatrix} = \cancel{t\ln t} + t - \cancel{t\ln t} = t$$

($\neq 0$ for $t \neq 0$).

3) (10a) $\Rightarrow$ $u_1(t) = -\int^t \frac{g(s)\, y_2(s)}{W(s)}\, ds$

$$= -\int^t \frac{\frac{1}{s}\cdot \cancel{s}\ln s}{\cancel{s}}\, ds = -\int^t \frac{\ln s}{s}\, ds \qquad v = \ln s, \quad dv = ds/s$$

$$= -\int^{s=t} v\cdot dv = -\frac{v^2}{2}\Big|^{s=t} = -\frac{(\ln s)^2}{2}\Big|^{s=t} = -\frac{(\ln t)^2}{2}.$$

(10b) $\Rightarrow$

$$u_2 = \int^t \frac{g(s)\, y_1(s)\, ds}{W(s)} = \int^t \frac{\frac{1}{s}\cdot\cancel{s}\cdot ds}{\cancel{s}} = \int^t \frac{ds}{s}$$

$$= \ln|s|\Big|^{s=t} = \ln|t| \xrightarrow[t>0]{\text{assume}} \ln t.$$

4) $$y_p = -\frac{(\ln t)^2}{2}\cdot t + \ln t \cdot t\ln t = \frac{1}{2}t(\ln t)^2$$

General sol'n:

$$y(t) = c_1 t + c_2 t\ln t + \frac{1}{2}t(\ln t)^2.$$

<u>Ex. 3</u>. Find y_p and then solve the IVP:

$$y'' + \omega^2 y = g(t).$$

$$y(0) = y_0, \quad y'(0) = y_0'.$$

<u>Sol'n:</u> 1) Lec. 10 (also Lec. 14) $\Rightarrow$

$y_1 = \cos\omega t$, $y_2 = \sin\omega t$ form a FS;
$W(t) = \omega$ (= const).

2) $u_1 = -\int_a^t \frac{g(s)\, y_2(s)}{W(s)}\, ds$ ← some constant.

Let us choose (a) so that $y_p(0) = 0$;
then c_1, c_2 in $y_c(t)$ will be chosen to satisfy the IC. ‖ details will be given later

Then $a = 0$, because $\int_0^{t=0} (\text{anything}) = 0$.

$$u_1(t) = -\int_0^t \frac{1}{\omega} \cdot g(s) \cdot \sin(\omega s)\, ds \equiv -\frac{1}{\omega} \int_0^t g(s) \sin(\omega s)\, ds$$

$$u_2(t) = \int_0^t \frac{1}{\omega}\, g(s) \cos(\omega s)\, ds.$$

$$(*) \quad y_p(t) = \frac{1}{\omega}\left[-\cos(\omega t) \cdot \int_0^t g(s) \sin(\omega s)\, ds + \sin(\omega t) \int_0^t g(s) \cos(\omega s)\, ds\right]$$

$$= \frac{1}{\omega} \int_0^t g(s) \left[\sin(\omega t) \cos(\omega s) - \sin(\omega s) \cdot \cos(\omega t)\right] ds$$

$$\overset{\text{trig identity}}{=} \frac{1}{\omega} \int_0^t g(s) \sin(\omega[t-s])\, ds.$$

General: $y(t) = c_1 \cos\omega t + c_2 \sin\omega t + \frac{1}{\omega} \int_0^t g(s) \sin(\omega[t-s])\, ds.$

3) $y(0) = y_0 \Rightarrow$

$$c_1 \cdot 1 + c_2 \cdot 0 + \underbrace{\cancel{y_p(0)}}_{0 \text{ (see above)}} = y_0 \Rightarrow c_1 = y_0.$$

$y'(0) = y_0' \Rightarrow$

$$c_1 \cdot (-\omega) \cdot 0 + c_2 \cdot \omega \cdot 1 + y_p'(0) = y_0'.$$

To find $y_p'(0)$, one could have differentiated Eq. (*) on previous page. For that, one needs:

- product rule, and
- Fund. Thm. of Calculus.

However, this is tedious.
Fortunately, there is another "Rule" of differentiation, which works here, as well as in many other circumstances. It is based on a composition of two familiar Rules.

- Familiar Rule 1:

$$\frac{d}{dt} \int_a^b f(t,s)\, ds = \int_a^b \frac{\partial f(t,s)}{\partial t}\, ds$$

- Familiar Rule 2 (Fund. Thm. of Calculus):

$$\frac{d}{dt} \int_a^t h(s)\, ds = h(t).$$

NEW RULE:

MUST MEMO-RIZE

$$\boxed{\frac{d}{dt} \int_a^t f(t,s)\, ds = \underbrace{\int_a^t \frac{\partial f(t,s)}{\partial t}\, ds}_{\text{from F. Rule 1}} + \underbrace{f(t,t)}_{\text{from F. Rule 2}}} \quad (14)$$

Apply this NEW RULE to $y_p'(t)$:

$$y_p'(t) = \cancel{\tfrac{1}{\omega}} \int_0^t g(s)\cdot \cancel{\omega}\cdot \cos(\omega[t-s])\,ds + \frac{1}{\omega}\cdot g(t)\cdot \sin(\underbrace{\omega[t-t]}_{0})$$

$$= \int_0^t g(s)\cos(\omega[t-s])\,ds.$$

Thus $y_p'(0) = \int_0^0 \ldots = 0$.

<u>Note 1</u>: With the $y_p(t) = \frac{1}{\omega}\int_0^t g(s)\sin(\omega[t-s])\,ds$, one has <u>convenient</u> conditions:

$$y_p(0) = 0 \text{ and } y_p'(0) = 0.$$

Thus, coming back to the top of p. 18-9:

$$y'(0) = y_0' \Rightarrow 0 + c_2\cdot\omega + 0 = y_0' \Rightarrow c_2 = \frac{y_0'}{\omega}.$$

Thus the solution of the IVP is:

$$y(t) = y_0\cos\omega t + \frac{y_0'}{\omega}\sin\omega t + \frac{1}{\omega}\int_0^t g(s)\sin(\omega[t-s])\,ds.$$

<u>Note 2</u>: An integral of the form

$$\int_0^t f(t-s)\,g(s)\,ds$$

is called a <u>convolution</u> of f and g. We will (hopefully) encounter it again in Chap. 5.

HW. Sec. 3.9

1 ← (g = const); compare to method of Lec. 15.16.

3, 4, 5, 7, 10, 13 ← find y_c, y_p, $y_{general}$.

16, 18 ← from given form $y = y_c + \int K(t-s)g(s)ds$ find α, β, y_0, y_0' in $y'' + \alpha y' + \beta y = g$.

WP: Solve IVP $y'' + \omega^2 y = G \sin \omega t$ using method of Ex. 3 from Notes. Hint: use $\sin a \cdot \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$.

Note for #4: Can do both int'ls w/Mathematica. (???)
If you want to do them by hand, use subst. $e^t = u$; then one of them will require a part. fr. expansion → look it up!

Note for #5: 2) Compare the so-obtained particular sol'n with that obtained by method from Lec. 17. How much do they differ by?
1) This will show you what happens to term $y_1 u_1$ (see Ex. 1 in class).

Note for #7: Again, compare your y_p with that from method of und. coeff.

Note for #13: 1) Denote $(t-1) = r$; relate dy/dt and dy/dr, and solve the entire problem in terms of $y(r)$.
2) Will need to find $y_2(r)$ by method of Lec. 13.

Note for #18: Use the form $y = c_1 y_1 + c_2 y_2 + y_1 u_1 + y_2 u_2$ and the expressions for u_1, u_2.
Answer: $\alpha = \beta = 0$, $y_0 = 0$, $y_0' = 1$.