Lecture 21. Higher-order linear DEs with constant coefficients.

1 Homogeneous DEs

Consider the homogeneous DE

$$y^{(n)} + a_{n-1}\cdot y^{(n-1)} + \dots + a_1 y' + a_0 y = 0, \quad (1)$$

where $a_{n-1}, \dots a_0 = \text{const}$.
The same substitution $y = e^{\lambda t}$ as for DE-2 yields the characteristic eq. for λ:

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0. \quad (2)$$

Solving it yields λ and hence $y = e^{\lambda t}$.

If all λ's are distinct, the set

$\{e^{\lambda_1 t}, e^{\lambda_2 t}, \dots e^{\lambda_n t}\}$ forms a FS for (1).

<u>Note</u>: This can be proved via Wronskian $\neq 0$ (an outline is in # 3.12.36 ← not a HW).
Intuitively, this can be argued from the fact that none of $e^{\lambda_j t}$ is a linear combination of $e^{\lambda_k t}$ for $\lambda_k \neq \lambda_j$, and hence all $e^{\lambda_j t}$ are lin. indep. (see Def.-A in Lec. 20, p. 20-8). And, we know that sol'ns in a FS are lin. independent (Thm. 3.8, Lec. 20).

Some differences from the case of DE-2 occur in regards to repeated roots.

$\boxed{1a}$ <u>Roots of (2) can have multiplicity more than two</u> (i.e., triple root, etc.)

<u>Ex. 1</u> Find the fundamental set of solutions of

$$y''' - 3y'' + 3y' - y = 0.$$

<u>Sol'n:</u> 1) Characteristic eqn.

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0 \Rightarrow$$

$$(\lambda - 1)^3 = 0 \Rightarrow \lambda_{1,2,3} = 1 \leftarrow \text{triple root}.$$

2) When λ was a double root, we showed in Lec. 13 that $y_1 = e^{\lambda t}$, $y_2 = te^{\lambda t}$ form a FS.
With a triple root, the situation is analogous:

$$y_1 = e^{\lambda t}, \quad y_2 = t e^{\lambda t}, \quad y_3 = t^2 \cdot e^{\lambda t}$$

form a FS. This can be shown by:

- verifying that y_2, y_3 are sol'ns of the DE, and
- verifying that their $W \neq 0$.

Thus, in our case $FS = \{e^t, te^t, t^2e^t\}$.

$\boxed{1b}$ Complex roots may be repeated

Ex. 2 Find the FS of sol's of

$$y^{(4)} + 8y'' + 16y = 0.$$

Sol'n: 1) Characteristic eqn:

$$\lambda^4 + 8\lambda^2 + 16 = 0.$$

We see that λ enters only via λ^2.
So denote $\lambda^2 = z$, $\Rightarrow$
the characteristic eqn. becomes:

$$z^2 + 8z + 16 = 0 \Rightarrow$$
$$(z+4)^2 = 0 \Rightarrow z_{1,2} = -4$$

$$\Rightarrow (\lambda^2 = -4)_{\text{repeated}} = \lambda_{1,2} = 2i, \ \lambda_{3,4} = -2i.$$

2) When the complex root is not repeated, we know from Lec. 14 that the sol'ns are
$y_1 = \cos 2t$, $y_3 = \sin 2t$.

When the complex root is repeated, one can show similarly to the repeated real root that the other pair of sol'ns is:

$$y_2 = t \cdot \cos 2t, \quad y_4 = t \cdot \sin 2t.$$

One can use the Wronskian to verify that they form a FS.
See Eq. (6) on p. 197 of book for complex roots of higher multiplicity.

2 Nonhomogeneous DEs

If we allow for a nonzero rhs $g(t) \neq 0$:

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = g(t), \quad (3)$$

then we can find a particular sol'n y_p using either the Method of Undetermined Coefficients (Lec. 17) — for const. a's and <u>some</u> $g(t)$, — or the Method of Variation of Parameters (Lec. 18) — for all $g(t)$ and even for nonconstant coefficients.

We will <u>not</u> consider here the Method of Variation of parameters because it will require some tricks that will appear misterious. (They will, however, become part of a straightforward algorithm when we study systems of linear DEs in Chap. 4.)

Here we will only consider an example of the Method of Undetermined Coefficients, whose idea is the same as that in Lec. 17.

<u>Ex. 3</u> Choose an appropriate form for a particular sol'n of

$$y^{(4)} - 3y''' + 3y'' - y' = t^2 + te^t + \sin t.$$

Sol'n: 1) Begin by finding the complementary sol'n.
The characteristic eqn. is: See Ex. 1

$$\lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda = 0 \Rightarrow$$

$$\lambda(\lambda-1)^3 = 0 \Rightarrow \lambda_1 = 0, \ \lambda_{2,3,4} = 1 \Rightarrow$$

$$y_c = c_1 + c_2 e^t + c_3 t e^t + c_4 t^2 e^t.$$

2) Particular sol'n has the general form

$$y_p = t^{r_1} \cdot (A_2 t^2 + A_1 t + A_0)$$

highest power of t^n in $g(t)$

$$+ t^{r_2} \cdot (B_1 t + B_0) e^t$$

highest power of t multiplying e^t in $g(t)$

$$+ t^{r_3} \cdot (D_1 \sin t + D_2 \cos t)$$

Here r_1, r_2, r_3 are to be chosen so that <u>no term</u> in y_p is part of y_c.

<u>r_1</u> If we chose $r_1 = 0$, $\Rightarrow$ A_0 is part of y_c. So we must choose $\boxed{r_1 = 1}$
(we always choose the <u>minimum</u> r).

<u>r_2</u> If we choose $r_2 = 0, 1,$ or 2, then the term $B_0 t^{r_2} e^t$ will be part of y_c.

Thus, we must choose $\boxed{r_2 = 3}$.

$\underline{r_3}$ Since $\sin t$ or $\cos t$ are not parts of y_c, $\Rightarrow$ choose $\boxed{r_3 = 0}$.

Thus,

$$y_p = t(A_2t^2 + A_1 t + A_0) + t^3(B_1 t + B_0)e^t + (D_1 \sin t + D_2 \cos t).$$

<u>HW</u>. <u>Sec. 3.12</u>.

3, 5, 9, 15, 17 $\leftarrow$ general hom. sol., or IVP.

21, 23, 25 $\leftarrow$ given y_c, find coeffs in DE.

<u>Sec. 3.13</u>

5, 7, 13, 14 $\leftarrow$ find y_c & y_p

15, 17, 18, 19 $\leftarrow$ form of y_p

31 $\leftarrow$ $y_c + y_p$, find C's in y_c that guarantee certain asymptotic behavior.

Answer for <u>14</u>: $c_1 + c_2 t + c_3 e^{-t} + 2t$

Ans. for <u>18</u>: