

## Lecture 22      Systems of 1st-order linear DEs : Introduction

### ① Motivation

We've mostly studied 1st-order & 2nd-order DEs for one variable  $y$ . They describe the rate of change or acceleration of one quantity:

$$\frac{dy}{dt} = \dots \quad \leftarrow \text{DE-1, rate of change}$$

$$\frac{d^2y}{dt^2} = \dots \quad \leftarrow \text{DE-2, acceleration}$$

However, in real world, things rarely change in isolation from one another. E.g., quantity  $y$  may depend on  $z$  and vice versa. Then we will have:

$$\frac{dy}{dt} = f_1(y, z)$$

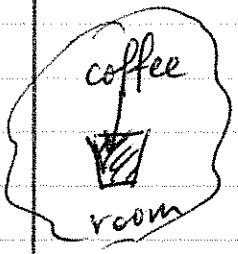
$$\frac{dz}{dt} = f_2(y, z)$$

Ex. 1 Set up a system of 1st-order DEs describing heating/cooling of a house with two "rooms".

### Preliminaries

In Lec. 1 (Ex. 2(b)) and HW 4 we

considered Newton's Cooling Law:



$$\frac{dT}{dt} = -K(T - T_{\text{room}}) = K(T_{\text{room}} - T)$$

This law is actually a corollary of the balance of heat (which is similar to balance equation for mixing problems in Lec. 4):

"amount of heat" (i.e., energy)

$$\frac{dQ}{dt} = \underbrace{\left( \text{Heat flow in} \right) - \left( \text{Heat flow out} \right)}_{m \cdot (T_{\text{external}} - T)} \quad (1)$$

some constant  $\rightarrow m \cdot (T_{\text{external}} - T)$

$T_{\text{ext}} > T \Rightarrow$  flow is in ( $> 0$ )

$T_{\text{ext}} < T \Rightarrow$  flow is out ( $< 0$ )

To supplement this equation, we need to relate "amount of heat" with temperature. From thermodynamics:

$$\Delta T = C \cdot \Delta Q$$

change in temperature caused by  $\Delta Q$

heat capacity (units =  $1^\circ\text{F}/\text{BTU}$ )

change in amount of heat (by adding or subtracting heat)

usually measured in BTU (British thermal unit = amount of heat needed to increase temp. of 1 lb of  $\text{H}_2\text{O}$  by  $1^\circ\text{F}$ ).

Then (1) becomes

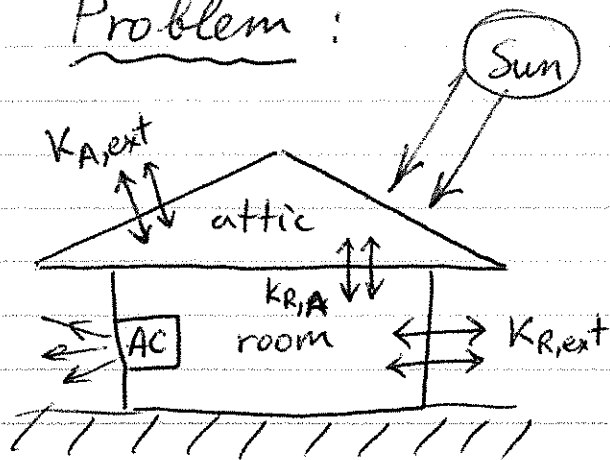
$$\frac{L}{c} \frac{dT}{dt} = m (T_{\text{ext}} - T)$$

$$\Rightarrow \frac{dT}{dt} = \underbrace{(cm)}_{\substack{\text{"K" of} \\ \text{Newton's} \\ \text{Cooling Law}}} \cdot (T_{\text{ext}} - T) \quad (2a)$$

or

$$\boxed{\frac{dQ}{dt} = \left(\frac{K}{c}\right) \cdot (T_{\text{ext}} - T)} \quad (2b)$$

Problem:



- Sun heats up the attic in a house at the rate of  $F_{\text{sun}}$  BTU/hr.

- The outside temperature is  $T_{\text{ext}} = 100^\circ\text{F}$ .

- The constants  $K$

in Newton's Law are:

- $K_{A,ext}$  — between Attic & Outside
- $K_{R,ext}$  — between Room & Outside
- $K_{R,A}$  — between Room & Attic

- The air conditioner in the room removes

$F_{AC}$  BTU/hr of heat.

- The heat capacity of the air (in room & attic) is  $C$ .

Set up a system of equations for the evolution of the temperatures  $T_A$  &  $T_R$  in the Attic and Room.

Sol'n:

Balance eq. for Attic:

$$\frac{dQ_A}{dt} = F_{sun} + \frac{K_{A,ext}}{C}(T_{ext} - T_A) + \frac{K_{R,A}}{C}(T_R - T_A)$$

↑
}
}

heat flow
heat exchange
heat exchange

from Sun
between
between

Attic & Outside
Attic & Room

Using  $dQ_A/dt = d(T_A/C)/dt$  we get:

$$\frac{dT_A}{dt} = C \cdot F_{sun} + K_{A,ext}(T_{ext} - T_A) + K_{R,A}(T_R - T_A) \quad (3a)$$

Balance for Room:

$$\frac{dQ_R}{dt} = -F_{AC} + \frac{K_{R,ext}}{C}(T_{ext} - T_R) + \frac{K_{R,A}}{C}(T_A - T_R)$$

↑
}
}

heat removal
heat exchange
heat exchange

by AC
between
between

Room & Outside
Room & Attic

Using  $dQ_R/dt = d(T_R/c)/dt$  we get:

$$\frac{dT_R}{dt} = -cF_{Ac} + K_{R,ext}(T_{ext} - T_R) + K_{R,A}(T_A - T_R) \quad (3b)$$

Eqs. (3a) & (3b) form the desired system of eqs.

Example 1 in book (Sec. 4.1) considers a mixing problem in two connected tanks, leading to a similar system.

In the most general case, a system of two linear, 1st-order DEs has the form:

$$\begin{aligned} y_1' &= p_{11}(t)y_1 + p_{12}(t)y_2 + g_1(t) \\ y_2' &= p_{21}(t)y_1 + p_{22}(t)y_2 + g_2(t) \end{aligned} \quad (4)$$

or in matrix form:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \quad (5a)$$

or, in vector-matrix notations:

$$\vec{y}' = P(t) \vec{y} + \vec{g}(t). \quad (5b)$$

For  $n$  lin. DEs, the system has the same form (5b).  
matrix



$$A \vec{x} \equiv [\vec{A}_1, \vec{A}_2, \vec{A}_3, \dots, \vec{A}_n] \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ = x_1 \vec{A}_1 + x_2 \vec{A}_2 + \dots + x_n \vec{A}_n$$

(6)

Ex. 2 Verify (6) for  $n=2$ .

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix} = x_1 \underbrace{\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}}_{\vec{A}_1} + x_2 \underbrace{\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}}_{\vec{A}_2}$$

✓

## 26 Calculus

- Derivatives — just differentiate each entry:

$$A' \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}' = \begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{pmatrix}.$$

Note: This should not be confused with the derivative of a determinant (see p. 20-5, Fact 4).

- Derivatives of sum and product — just as for scalars (p. 219).

3. Integration - just integrate each term.

Ex. 3 Find  $\int A(t)dt$ ,  $A(t) = \begin{pmatrix} e^{3t} & 2t \\ 0 & -1 \end{pmatrix}$ .

Sol'n:  $\int A(t)dt = \begin{pmatrix} \int e^{3t} dt & \int 2t dt \\ \int 0 dt & \int -1 dt \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{3}e^{3t} + C_{11} & t^2 + C_{12} \\ 0 + C_{21} & -t + C_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{3}e^{3t} & t^2 \\ 0 & -t \end{pmatrix} + \underbrace{\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}}$

The integration  $\nearrow$   
"constant" becomes  
a constant matrix

HW. Sec. 4.1. 1, 2, 4  $\leftarrow$  simple operations  
7  $\leftarrow$  inversion of a  $2 \times 2$

16, 17, 19, 21  $\leftarrow$  integration

25  $\leftarrow$   $(A^2)' \neq 2AA'$ , Hint: Take  $A = \begin{pmatrix} a(t) & b \\ 0 & c(t) \end{pmatrix}$   $\leftarrow$  const

Ans. #4:  $t^2(-t-1)$

#16:  $\begin{pmatrix} -t + C_1 \\ t^2 + C_2 \end{pmatrix}$

EC #1 #27 + 28 (about  $(A^{-1})'$ )  $\leftarrow$  together

#30 (+ read Ex. 1 on pp. 214-215).