Lecture 23 General properties of linear systems of first-order DEs

We'll follow the outline of Lecture 20, where we listed general properties of higher-order linear DEs for 1 variable.

In fact, the outline of topics 1–3 will mimic that of Lec. 20. But, before we get to topic 1, we'll establish a relation between higher-order DEs for 1 variable and systems of 1-st order DEs.

0 n th order scalar DEs as 1st-order linear systems

Ex. 1 Rewrite the 2nd-order DE

$$y'' - e^t \cdot y' + 3y = \sin 2t$$

as a 1st-order linear system.

Sol'n: Let $y_1 = y$, $y_2 = y'$.

First eq. of the system is simply $y_1' = y_2$
$(y' = y')$. ✓

Second eq. of the system will come from the given DE:

$$y'' = e^t \cdot y' - 3y + \sin 2t$$

$$\underbrace{(y')'}_{y_2} = e^t \cdot \underbrace{(y')}_{y_2} - 3 \cdot \underbrace{y}_{y_1} + \sin 2t$$

$$y_2' = -3\cdot y_1 + e^t\cdot y_2 + \sin 2t.$$

Putting them together:

$$y_1' = 0\cdot y_1 + 1\cdot y_2$$
$$y_2' = -3\cdot y_1 + e^t\cdot y_2 + \sin 2t \quad \Rightarrow$$

$$\underbrace{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}'}_{\vec{y}\,'} = \underbrace{\begin{pmatrix} 0 & 1 \\ -3 & e^t \end{pmatrix}}_{\substack{P(t) \\ 2\times 2 \text{ matrix}}} \underbrace{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}_{\vec{y}} + \underbrace{\begin{pmatrix} 0 \\ \sin 2t \end{pmatrix}}_{\vec{g}(t)}.$$

Ex. 2 Rewrite the 3rd-order DE

$$y''' + p_2(t)\,y'' + p_1(t)\,y' + p_0(t)\,y = g(t)$$

as a 1st-order linear system.

Sol'n: Let $y_1 = y$, $y_2 = y'$, $y_3 = y''$.

First & second eqs. of the system are simply:

$$y_1' = y_2 \quad \text{and} \quad y_2' = y_3$$
$$(y' = y') \qquad\qquad ((y')' = y'').$$

Third eq. of the system comes from the given DE:

$$y''' = -p_2\cdot y'' - p_1\cdot y' - p_0\, y + g(t)$$
$$\underbrace{(y'')'}_{y_3'} = -p_0\cdot \underbrace{y}_{y_1} - p_1\cdot \underbrace{y'}_{y_2} - p_2 \underbrace{y''}_{y_3} + g(t)$$

$$y_3' = -p_0 y_1 - p_1 y_2 - p_2 y_3 + g(t).$$

Putting them together:

$$y_1' = 0\cdot y_1 + 1\cdot y_2 + 0\cdot y_3$$

$$y_2' = 0\cdot y_1 + 0\cdot y_2 + 1\cdot y_3 \qquad \Rightarrow$$

$$y_3' = -p_0 y_1 - p_1 y_2 - p_2 y_3 + g(t).$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -p_0 & -p_1 & -p_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}.$$

So, same general form

$$\vec{y}\,' = P(t)\,\vec{y} + \vec{g}(t) \qquad (1)$$

$n\times 1$ vector $\qquad$ $n\times n$ matrix $\qquad$ $n\times 1$-vector.

<u>Q:</u> What do we achieve by rewriting a scalar higher-order DE as a lin. system?

<u>A:</u>
- unity of the approach to both these types of problems
- method of variation of parameters for nonhomogeneous DEs will become transparent
- unity of approach to <u>numerical</u> solution of both problems (this will <u>not</u> be considered in this course).

1 Existence and uniqueness

Thm. 4.1 Consider an IVP that consists of the DE (system) (1) and an initial condition

$$\vec{y}(t_0) = \vec{y}_0 . \qquad (2)$$

Let the n^2 coefficients $p_{ij}(t)$ of matrix $P(t)$ and n coefficients $g_1(t), \dots g_n(t)$ of vector $\vec{g}$ be continuous on $t \in (a, b)$, and let $t_0 \in (a, b)$.
Then the IVP has a unique sol'n on (a, b).

In what follows we assume that conditions of this E&U thm. always hold.

2 Properties of homogeneous linear systems

2a Principle of linear superposition

Note about notations: In topic 1, subscripts in $y_1, y_2, \dots$ meant components of $\vec{y}$. Below, subscripts label different vectors $\vec{y}_1, \vec{y}_2, \dots$

Thm. 4.2 Let $\vec{y}_1, \dots \vec{y}_r$ be solutions of the homogeneous linear DE system

$$\vec{y}\,' = P(t)\, \vec{y} . \qquad (3)$$

Then their linear combination

$$\vec{y} = c_1 \vec{y}_1 + \dots + c_r \vec{y}_r ,$$

where $c_1, \dots c_r$ are any constants, is also a solution of (3).

Proof: book, p. 229 (same idea as before: Lec. 2, Lec. 20).

$\boxed{2b}$ Fundamental sets of sol'ns, and Wronskian

Let $\vec{y_1}, \dots \vec{y_n}$ be n solutions of the linear system (3) of n DEs.
Set $\{\vec{y_1}, \dots \vec{y_n}\}$ is called a FS of sol'ns if any sol'n of IVP (3)+(2) can be written as

$$\vec{y} = c_1\vec{y_1} + \dots + c_n\vec{y_n} \qquad (4)$$

for some constants $c_1, \dots c_n$.

(Somewhat conversely, if $\{\vec{y_1}, \dots \vec{y_n}\}$ is a FS, then (4) is called the general sol'n of (3).)

Let us now determine a condition which determines whether a given set of sol'ns is a FS or not.

Suppose $\{\vec{y_1}, \dots \vec{y_n}\}$ is some set of sol'ns of (3)

Let's try to construct a sol'n of IVP (3)+(2) in the form (4). All we need is to satisfy the IC (2), because we know that (4) satisfies the DE (3). So:

$$c_1\vec{y_1}(t_0) + c_2\vec{y_2}(t_0) + \dots + c_n\vec{y_n}(t_0) = \vec{y_0} \qquad (5)$$

Using the VERY IMPORTANT PROPERTY (6) from Lec. 22 (p. 22-7):

$$[\vec{A_1}, \vec{A_2}, \dots \vec{A_n}]\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{A_1} + x_2\vec{A_2} + \dots + x_n\vec{A_n},$$

we rewrite (5) as:

$$\underbrace{[\vec{y}_1(t_0), \vec{y}_2(t_0), \dots \vec{y}_n(t_0)]}_{\text{let's call this matrix}} \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}}_{\vec{c}\ \leftarrow\ n\times 1 \text{ vector.}} = \vec{y}_0 \quad (6a)$$

$\Psi(t_0)$

"psi"

Then, (6a) can be rewritten as

$$\Psi(t_0)\,\vec{c} = \vec{y}_0 . \qquad (6b)$$

Then:

- We know that the sol'n of IVP (3)+(2) exists by the E & U theorem (topic 1).
- This is equivalent to the fact that the constants $c_1, \dots c_n$ in (5), or vector $\vec{c}$ in (6b), can be found for <u>any</u> $\vec{y}_0$.
- From lin. algebra we know that this can be guaranteed by $\Psi(t_0)$ being nonsingular, or

$$W(t) \equiv \boxed{\det \Psi(t_0) \neq 0.} \qquad (7)$$

Wronskian

condition for $\{\vec{y}_1, \dots \vec{y}_n\}$ to form a FS.

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<u>Ex. 3(a)</u> Verify that $\vec{y}_1 = \begin{pmatrix} t \\ 1 \end{pmatrix}$, $\vec{y}_2 = \begin{pmatrix} t^{-1} \\ -t^{-2} \end{pmatrix}$ form a FS of sol'ns for

$$\vec{y}\,' = \begin{pmatrix} 0 & 1 \\ t^{-2} & -t^{-1} \end{pmatrix} \vec{y}, \quad t > 0.$$

<u>Sol'n</u>: 1) We first need to check that $\vec{y_1}, \vec{y_2}$ are solutions. Do this for $\vec{y_1}$; for $\vec{y_2}$ it is similar.

<u>lhs</u>: $\vec{y_1}' = \begin{pmatrix} t \\ 1 \end{pmatrix}' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

<u>rhs</u>: $\begin{pmatrix} 0 & 1 \\ t^{-2} & -t^{-1} \end{pmatrix}\begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} 0+1 \\ t\cdot t^{-2} - t^{-1}\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

lhs = rhs. ✓.

2) To verify if they form a FS, we need to find their Wronskian at some point $t>0$. The easiest point is $t_0 = 1$:

$$W(t_0=1) = \begin{vmatrix} 1 & 1^{-1} \\ 1 & -1^{-2} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0,$$

$\Rightarrow \{\vec{y_1}, \vec{y_2}\}$ form a FS.

<u>Note</u>: This lin. system is just a 2nd-order DE in disguise:

$$\begin{aligned} y_1' &= 0\cdot y_1 + 1\cdot y_2 & \Rightarrow\quad & y_1' = y_2 \\ y_2' &= \tfrac{1}{t^2} y_1 - \tfrac{1}{t} y_2 & & (y_1')' = \tfrac{1}{t^2} y_1 - \tfrac{1}{t}(y_1)' \end{aligned}$$

$$\Rightarrow \quad y_1'' + \tfrac{1}{t} y_1' - \tfrac{1}{t^2} y_1 = 0.$$

(Compare with Ex. 1 & Ex. 2.)

<u>Ex. 3(b)</u> Find the solution of that lin. system satisfying the initial cond.

$$\vec{y}(1) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

<u>Sol'n:</u> We follow the steps by which we obtained the Wronskian of a lin. system (see Eq. (7)).

General sol'n: $\vec{y} = c_1 \vec{y_1} + c_2 \vec{y_2} = [\vec{y_1}, \vec{y_2}]\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$\vec{y}(1) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow [\vec{y_1}(1), \vec{y_2}(1)]\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

We exhibited this matrix in Ex. 3(a), $\Rightarrow$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

formula for a 2×2 A^{-1} on p. 217

<u>Note</u>: It is a coincidence that A^{-1} and A look proportional.
In general they are not! Again, see p. 217.

$$= -\frac{1}{2}\begin{pmatrix} -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/2 \\ -1/2 \end{pmatrix}.$$

Thus, $\vec{y}(t) = \frac{5}{2}\vec{y_1}(t) - \frac{1}{2}\vec{y_2}(t)$

$$= \frac{5}{2}\begin{pmatrix} t \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} t^{-1} \\ -t^{-2} \end{pmatrix}.$$

We will now state a counterpart of the Abel Thm. for $W(t)$ of a scalar DE:

$$W'(t) = -p_{n-1}(t)\, W(t).$$

For lin. system, this is called Liouville's formula

First, we need a definition.

Def: Let
$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & & \ddots & \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}.$$

Then trace of P is:

$$tr[P] = p_{11} + p_{22} + p_{33} + \ldots + p_{nn}$$ (sum of diagonal elements).

Thm. 4.4 (Liouville's formula).

Let $\{\vec{y}_1, \ldots \vec{y}_n\}$ be a set of sol'ns of lin. system (3) ($\vec{y}\,' = P\vec{y}$) and let W be the Wronskian of these solutions. Then

$$W'(t) = tr[P] \cdot W(t). \qquad (8a).$$

Solving (8a), we get:

$$W(t) = W(t_0) \cdot e^{\int_{t_0}^{t} tr[P(s)]\, ds}. \qquad (8b)$$

Thus, as before, the Wronskian is either 0 or nonzero for all t in the interval of existence and uniqueness of the solution.

See pp. 23-9(a-c) posted after this Lecture (Proof of Abel's Thm.).

Note: For scalar DEs of order n, Liouville's formula reduces to Abel's Thm.
E.g., in Ex. 2 we showed that

$$y''' + p_2 y'' + p_1 y' + p_0 y = 0 \qquad \text{(DE-3)}$$

is equivalent to

$$\vec{y}\,' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -p_0 & -p_1 & -p_2 \end{pmatrix} \vec{y}, \quad \vec{y} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}. \quad \text{(lin. sys.)}$$

For (DE-3), Abel's Thm. yields

$$W'(t) = -p_2(t)\, W.$$

For (lin. sys.), $tr[P] = 0 + 0 + (-p_2)$, $\Rightarrow$
Liouville's formula yields

$$W'(t) = (-p_2)\, W,$$

which is indeed the same.

Proof of Liouville's formula for $n=2$ is in ECH.

$\boxed{2c}$ FS of solutions of lin. systems always exist.
(a.k.a. how to find a FS).

The idea is the same as for scalar DEs.

Consider n IVPs which share the same lin. system (3):

$\vec{y}' = P\vec{y}$ (for any $\vec{y}$ from $\{\vec{y_1}, \dots \vec{y_n}\}$)

but different initial conditions, which we illustrate for $n=3$:

$$\vec{y_1}(t_0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{y_2}(t_0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{y_3}(t_0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Each of these solutions exist by the E&U Thm. Moreover, their Wronskian at t_0 is:

$$W(t_0) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0,$$

$\Rightarrow$ by Liouville's formula $W(t) \neq 0$ in the interval of E & U.

$\boxed{2d}$ Solutions in a FS are lin. independent

If $\{\vec{y_1}, \dots \vec{y_n}\}$ is a FS, then the fund. matrix $\Psi(t) = [\vec{y_1}, \dots \vec{y_n}]$ is nonsingular because $\det \Psi = W \neq 0$. But it is known from lin. algebra that in any nonsingular matrix, columns ($\vec{y_1}, \dots \vec{y_n}$ in this case) are lin. independent.

$\boxed{2e}$ Relation between two FSs

Let $\Psi = [\vec{y_1}, \dots \vec{y_n}]$ and $\hat{\Psi} = [\hat{\vec{y_1}}, \dots \hat{\vec{y_n}}]$ be fundamental matrices of the same lin. system (3). Then:

<u>Thm. 4.5</u> (a) There exists a <u>unique</u> matrix C s.t.

$$\hat{\Psi} = \Psi C. \qquad (9)$$

(b) and C is nonsingular.

<u>Proof (for $n=2$)</u>

(a) Since $\{\vec{y}_1, \vec{y}_2\}$ is a FS, $\Rightarrow$ by the <u>linear superposition principle</u> (topic 2 $\boxed{a}$)

$$\hat{\vec{y}}_1 = c_{11}\vec{y}_1 + c_{21}\vec{y}_2 = [\vec{y}_1, \vec{y}_2]\begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix}$$

<u>some constants</u>

$$= \Psi\begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix}.$$

Similarly, $\hat{\vec{y}}_2 = c_{12}\vec{y}_1 + c_{22}\vec{y}_2 = \Psi\begin{pmatrix} c_{12} \\ c_{22} \end{pmatrix}$.

Thus, $[\hat{\vec{y}}_1, \hat{\vec{y}}_2] = \left[\Psi\begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix}, \Psi\begin{pmatrix} c_{12} \\ c_{22} \end{pmatrix}\right]$

by Fact 4 on p. 22-6

$$\hat{\Psi} = \Psi C. \quad ✓.$$

(b) $\underbrace{\det \hat{\Psi}}_{\neq 0} = \det(\Psi C) = \underbrace{\det \Psi}_{\neq 0} \cdot \det C$

$\Rightarrow \det C \neq 0 \Rightarrow C$ is nonsingular.

<u>How to find this C?</u>

$$\hat{\Psi} = \Psi \cdot C \Rightarrow \Psi^{-1}\hat{\Psi} = \overbrace{\Psi^{-1}\Psi}^{I} \cdot C \Rightarrow \boxed{\Psi^{-1}\hat{\Psi} = C}$$

(3) Nonhomogeneous linear systems of DEs

$$\vec{y}\,' = P\vec{y} + \vec{g} \qquad (1).$$

As before:

$$\begin{pmatrix}\text{General}\\ \text{sol'n of}\\ \text{nonhom. system}\end{pmatrix} = \begin{pmatrix}\text{General}\\ \text{sol'n of}\\ \text{homog. system}\end{pmatrix} + \begin{pmatrix}\text{Particular}\\ \text{sol'n of}\\ \text{nonhom. system}\end{pmatrix}$$

$$\vec{y} = \vec{y_c} + \vec{y_p}.$$

We also have a superposition principle for nonhomogeneous systems:

Thm. 4.8 (Sec. 4.8). Let $\vec{v_1}$ solve (1) with $\vec{g}=\vec{g_1}$ and $\vec{v_2}$ solve (1) with $\vec{g}=\vec{g_2}$. Then $(a_1\vec{v_1}+a_2\vec{v_2})$ solves (1) with $\vec{g}=a_1\vec{g_1}+a_2\vec{g_2}$, where a_1, a_2 are any constants.

MW. Sec. 4.2
- #7 ← rewrite as $\vec{y}\,'=A\vec{y}$.
- #11, 13, ← scalar DE to lin. sys.
- 15, 16 ← lin. sys. as scalar DE
- 19, 21 ← 2 DE-2s as lin. sys.
- #9 ← rewrite as $\vec{y}\,'=A\vec{y}$, verify sol'n; solve IVP.

This is from Sec. 4.2

Sec. 4.3
- #1, 3 ← rewrite as $\vec{y}\,'=P\vec{y}$, verify sol'n.
- #7, 10 ← verify if a FS
- #15, 17, 23 ← verify if FS, solve IVP.
- #25, 26, 28 ← Liouville's Thm.
- #29, 30, 31 ← all (b, c) ← $\hat{\Psi}=\Psi C$; #33 ← similar.

Sec. 4.8. #11 given $P, \vec{y}$, find $\vec{g}$.