Here we will prove the special case of Liouville's Theorem, called _Abel's Thm_ ; also, we'll do so only for $n=2$. The _general case of Abel's Thm_. is stated in _Thm. 3.6_.

Abel's Theorem for $n=2$

Let u, v be two solutions of a (DE-2):

$$y'' + p(t) y' + q(t) y = 0. \qquad (9a-1)$$

From Lec. 11, topic ②⑧, their Wronskian is:

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} \qquad (9a-2)$$

Then one has:

$$\boxed{W'(t) = -p(t) W(t)} \qquad (9a-3)$$

or, equivalently :

$$\boxed{W(t) = W(t_0) \cdot \exp\left[-\int_{t_0}^{t} p(s)\,ds\right]} \qquad (9a-4)$$

Proof :

__Step 1__ : Rewrite (DE-2) $(9a-1)$ as a system of first-order DEs.

By topic ⓪ of this Lecture, denote $y_1 = y$, $y_2 = y'$; then $(9a-1)$ becomes (see Ex. 1 & 2) :

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -p\,y_2 - q\,y_1 \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad (9a-5)$$

So, if u, v are two solutions of $(9a-1)$,

then $\quad \vec{u} = \begin{pmatrix} u \\ u' \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v \\ v' \end{pmatrix}$

are two solutions of the equivalent system $(9a-5)$,
and their Wronskian is, by Eq. (7) on p. 23-6:

$$W = \det \begin{vmatrix} u & v \\ u' & v' \end{vmatrix},$$

which is the same as the definition $(9a-2)$ for $(9a-1)$.

Now, the matrix P defined on p. $(23-9)$, is:

$$P = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \Rightarrow tr[P] = 0 + (-p) = -p \qquad (9b-1)$$

Using $(9b-1)$, one can easily verify that the Liouville Thm 4.4 (formula $(8a)$ on p. 23-9) reduces to Abel's Thm. $(9a-3)$.

Step 2   Since a Wronskian is a determinant, we need some auxiliary facts about determinants (stated below for $n=2$, but true in general):

Fact 1 $\begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix} = 0$

(if two rows or columns in a matrix are the same, det $=0$).

Fact 2 $\begin{vmatrix} a_1 & a_2 \\ k b_1 & k b_2 \end{vmatrix} = k \cdot \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

$\boxed{\text{Fact 3}}$

$$\begin{vmatrix} a_1 & a_2 \\ b_1+c_1 & b_2+c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$$

$\boxed{\text{Fact "2+3"}}$

$$\begin{vmatrix} a_1 & a_2 \\ kb_1 + mc_1 & kb_2 + mc_2 \end{vmatrix} = K \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + m \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$$

$\boxed{\text{Fact 4}}$ (product rule for determinants)

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}' = \begin{vmatrix} a_1' & a_2' \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ b_1' & b_2' \end{vmatrix}$$

$$\left(e.g., \; (a_1 b_2)' = a_1' b_2 + a_1 b_2' \right)$$

Step 3   Proof of $(9a-3)$

Differentiate the Wronskian $(9a-2)$:

$$\mathbf{W}' = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}' \overset{\boxed{F4}}{=} \underbrace{\begin{vmatrix} u' & v' \\ u' & v' \end{vmatrix}}_{=0 \text{ by } \boxed{F1}} + \begin{vmatrix} u & v \\ u'' & v'' \end{vmatrix}$$

$\overset{\text{by the DE } (9a-1)}{=} \begin{vmatrix} u & v \\ (-pu'-qu) & (-pv'-qv) \end{vmatrix} \overset{F "2+3"}{=}$

$$= -p \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} - q \cdot \underbrace{\begin{vmatrix} u & v \\ u & v \end{vmatrix}}_{0 \text{ by } \boxed{F1}} = -\mathbf{p \cdot W}$$

This proves $(9a-3)$.