Lecture 24. Homogeneous linear systems with constant coefficients

① The eigenvalue problem

We consider the linear system

$$\vec{y}\,' = A\,\vec{y}\,, \qquad (1)$$

where all entries of the $n \times n$ matrix A are const.
We seek its solution in the form

$$\vec{y} = e^{\lambda t} \cdot \vec{x} \qquad (2)$$

$\vec{x}$ ← constant vector

↑ need a vector here because $\vec{y}$ is a vector

Substitute (2) into (1):

$$\lambda \cancel{e^{\lambda t}}\,\vec{x} = A\vec{x} \cdot \cancel{e^{\lambda t}}, \text{ or}$$

$$\boxed{A\vec{x} = \lambda\,\vec{x}, \quad \vec{x} \neq \vec{0}.} \qquad (3)$$

Eq. (3) is the eigenvalue problem, and requires the knowledge of both λ and $\vec{x}$.

λ = eigenvalue, ("eigen" = "own", or
$\vec{x}$ = eigenvector. "characteristic").
$(\lambda, \vec{x})$ = eigenpair of matrix A.

Ex. 1 Find the eigenpairs of

$$A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}.$$

__Sol'n:__ Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then Eq. (3) $\Rightarrow$

1) __Find λ:__ $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow$

$$\underbrace{\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}}_{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \underbrace{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}}_{\lambda I}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

I ← ~~ide~~ identity matrix

$$\underbrace{\begin{pmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{pmatrix}}_{(A-\lambda I)}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$(A - \lambda I)\,\vec{x} = \vec{0} \qquad (4)$$

$\Rightarrow$ (by definition!) $\boxed{A - \lambda I \text{ is singular}} \Leftrightarrow$

$$\det(A - \lambda I) = 0 \qquad (5)$$

$$\begin{vmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$(4-\lambda)(1-\lambda) + 2 = 0 \Rightarrow \lambda^2 - 5\lambda + 6 = 0$$

$$\Rightarrow (\lambda - 3)(\lambda - 2) = 0 \Rightarrow \lambda_{1,2} = 2, 3.$$

2) __Find eigenvectors.__

__$\lambda = 2$__ $\begin{pmatrix} 4-2 & -2 \\ 1 & 1-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$2x_1 - 2x_2 = 0 \Rightarrow x_2 = x_1.$$

A convenient choice is $x_1 = 1$; then by the above, $x_2 = x_1 = 1$. Thus, the first eigenvector is:

$$\vec{x_1} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

<u>Note 1</u>: We could have chosen $x_1 = a =$ <u>any number except 0</u>; then $x_2 = a$, and

$$\vec{x_1} = \begin{pmatrix} a \\ a \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, we can multiply an eigenvector by <u>any nonzero number</u>, and the result <u>is still an eigenvector</u>! Indeed:

$$a \cdot (A\vec{x} = \lambda \vec{x})$$

$$aA\vec{x} = a\lambda\vec{x}$$

$$A(a\vec{x}) = \lambda(a\vec{x}) \Rightarrow a\vec{x} \text{ is an eigenvector}$$

<u>Note 2</u>: An eigenvector $\neq \vec{0}$ by definition. (So, above, $a \neq 0$.)
However, an <u>eigenvalue</u> can $= 0$:
$\lambda = 0$ simply means that A is singular:

$$\det(A - 0 \cdot I) = \det A = 0.$$

Continuing with the example:

$\underline{\lambda = 3} \Rightarrow \begin{pmatrix} 4-3 & -2 \\ 1 & 1-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$

$$\begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 - 2x_2 = 0.$$

For convenience, take $x_1 = 2$, $\Rightarrow x_2 = \frac{1}{2}x_1 = 1$. Then

$$\vec{x_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Ex. 2 Find eigenpairs of

$$A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \quad \alpha \neq 0.$$

Sol'n: 1) Find eigenvalues:

$$\begin{vmatrix} 1-\lambda & \alpha \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 - 0 \cdot \alpha = 0 \Rightarrow (\lambda - 1)^2 = 0 \Rightarrow \lambda_{1,2} = 1.$$

We have a repeated root!

2) Find eigenvectors:

$\lambda = 1$

$$\begin{pmatrix} 1-1 & \alpha \\ 0 & 1-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} 0 \cdot x_1 + \alpha \cdot x_2 = 0 \\ 0 = 0 \end{matrix} \Rightarrow$$

$\alpha x_2 = 0 \Rightarrow x_2 = 0$ (since $\alpha \neq 0$).

No information about $x_1 \Rightarrow x_1 =$ any number.

So, for convenience, take $x_1 = 1$. Then

$$\vec{x_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

And we do not have a second eigenvector!

<u>Note</u>: We can multiply $\vec{x_1}$ by any number, e.g., have $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$,
but this will <u>not</u> be a new eigenvector: see Note 1 in Ex. 1.

Thus, in this Ex. 2, we have a repeated eigenvalue and <u>just one</u> eigenvector.

<u>Ex. 3</u> Find the eigenpairs of

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

<u>Sol'n:</u> 1) Find eigenvalues:

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = 0$$

A calculation (verified by Mathematica) yields:

$$-\lambda^3 + 6\lambda^2 - 9\lambda = 0 \quad \Rightarrow$$
$$-\lambda(\lambda^2 - 6\lambda + 9) = 0 \quad \Rightarrow$$
$$\lambda(\lambda - 3)^2 = 0 \quad \Rightarrow \lambda_1 = 0, \ \lambda_{2,3} = 3.$$

2) <u>Find eigenvectors:</u>

<u>$\lambda = 0$</u>

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving this by transformation to Reduced Echelon Form yields: (a.k.a. Gaussian elimination)

You are not required to use this exact algorithm, but you ARE required to be able to solve a 3x3 linear system.

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

"free", i.e. arbitrary

$\Rightarrow x_1 - x_3 = 0 \;\&\; x_2 - x_3 = 0 \Rightarrow x_1 = x_3,\ x_2 = x_3.$

Let $x_3 = 1 \Rightarrow x_1 = 1 \Rightarrow x_2 = 1 \Rightarrow$

$\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$ (choosing another $x_3 \neq 0$ will yield a proportional $\vec{x_1}$)

$\underline{\lambda = 3}$ $\quad (A - 3I)\vec{x} = \vec{0} \Rightarrow$

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

All three eqs. are the same, $\Rightarrow$ we just have one:

$$x_1 + x_2 + x_3 = 0, \quad \Rightarrow \quad x_1 = -x_2 - x_3,$$

where now both x_2 & x_3 are "free" variables.

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 \\ 1 \cdot x_2 + 0 \cdot x_3 \\ 0 \cdot x_2 + 1 \cdot x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} x_3.$ Considering two pairs: $(x_2 = 1, x_3 = 0)$ and $(x_2 = 0, x_3 = 1)$ yields

Thus: $\vec{x_2} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{x_3} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Note 1: This choice is not unique, but any choices will yield exactly two independent vectors.

Note 2: As in Ex. 2, we've had a repeated eigenvalue, but to it there correspond two distinct eigenvectors. Thus, this 3x3 matrix A has a full set (i.e. $3 = n$) of eigenvectors.

Two sufficient (but not necessary!) criteria when $n \times n$ matrix A has a full set (i.e. n) of eigenvectors:

$\boxed{1}$ If A is real and symmetric, then it has a full set of eigenvectors.
(In this case all λ's are also real.)

$\boxed{2}$ If A has n distinct eigenvalues, i.e. $\lambda_i \neq \lambda_j$ for $i \neq j$, then A has a full set of eigenvectors.

(2) Fundamental set of solutions and solving an IVP.

Thm. 4.6 Consider the homogeneous lin. system with a constant $n \times n$ matrix A. Let A has n eigenpairs $(\lambda_1, \vec{x_1}), \dots (\lambda_n, \vec{x_n})$, where all $\vec{x_i}$ are linearly independent. (In other words, A has a full set of eigenvectors.)
Then

$$\{ e^{\lambda_1 t} \vec{x_1}, e^{\lambda_2 t} \vec{x_2}, \dots, e^{\lambda_n t} \vec{x_n} \}$$

is a FS.

Note: We require n lin. independent eigenvectors, but some of eigenvalues may be repeated (as in Ex. 3).

Proof of Thm. 4.6: At $t=0$,

$W(0) = \det[\vec{x_1}, \vec{x_2}, \dots \vec{x_n}] \neq 0$ because $\vec{x_1}, \dots \vec{x_n}$ are lin. independent. Thus, by Liouville's formula (Thm. 4.4), $W(t) \neq 0$ $\Rightarrow$ this is a FS.

Ex. 4 Solve the IVP

$$\vec{y}\,' = A\vec{y}, \quad \vec{y}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

where $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$ as in Ex. 1.

Sol'n: By Thm. 4.6 and Ex. 1,

$$\{e^{\lambda_1 t}\vec{x_1}, e^{\lambda_2 t}\vec{x_2}\} = \{e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\} \text{ is a FS.}$$

Then we seek $\vec{y}(t) = e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot c_1 + e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} c_2$.
Requiring $\vec{y}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ yields:

$$c_1 \cdot 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \cdot 1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\Downarrow$ Eq.(6) on p. 22-7

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \xRightarrow{\text{solving by any method}}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

Thus, $\vec{y}(t) = 3e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$

Ex. 5 Same, but $A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, $\alpha \neq 0$, as in Ex. 2.

Sol'n: In Ex. 2 we've found only one eigenvector, $\Rightarrow$ we do not yet have a FS.

Such a FS can be found but it requires a concept beyond that of eigenvectors.

It is considered in Sec. 4.7 of textbook, but we will not consider it in the course.

Since we do not know a FS in this problem, we cannot write

$$\vec{y}(t) = c_1 \vec{y_1} + c_2 \vec{y_2}$$

and hence cannot solve the IVP.

Ex. 6 Same, but $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$ and $\vec{y}(0) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Sol'n: By Thm. 4.6 and Ex. 3,

$$\{e^{\lambda_1 t}\vec{x_1}, e^{\lambda_2 t}\vec{x_2}, e^{\lambda_3 t}\vec{x_3}\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, e^{3t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, e^{3t}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

is a FS. Then seek

$$\vec{y}(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{3t}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Note that even though $\lambda = 3$ is a repeated eigenvalue, we have a full set of eigenvectors for A $\Rightarrow$ we have a FS for the lin. system.

From the given initial condition we have:

$$c_1\begin{pmatrix}1\\1\\1\end{pmatrix} + c_2\begin{pmatrix}-1\\0\\1\end{pmatrix} + c_3\begin{pmatrix}-1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} \Rightarrow$$

$$\begin{pmatrix}1 & -1 & -1\\1 & 0 & 1\\1 & 1 & 0\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} \Longrightarrow \text{solving by any method}$$

$$c_1 = 2,\ c_2 = 1,\ c_3 = 0$$

$$\Rightarrow \vec{y}(t) = 2\begin{pmatrix}1\\1\\1\end{pmatrix} + e^{3t}\begin{pmatrix}-1\\0\\1\end{pmatrix}.$$

(3) Complex eigenvalues

When an eigenvalue λ of A is complex, then so will be the corresponding eigenvector $\vec{x}$ Indeed

$$\underset{\text{real}}{A}\,\vec{x} = \underset{\text{complex}}{\lambda}\,\vec{x}$$

and the two sides of the equation can match only when $\vec{x}$ is complex, not real.

Thus, we will have a complex solution

$\vec{y} = e^{\lambda t}\vec{x}$ in our FS. However, if we ar solving a problem with a real matrix A, we want the solutions to also be real. How do we obtain such real solutions from complex ones?

<u>Thm. 4.7</u> Consider the lin. system (3) with a <u>real</u> matrix A. Let $\vec{y} = \vec{u} + i\vec{v}$ (where $\vec{u}, \vec{v}$ are real) be a complex solution of this DE system. Then each of $\vec{u}$ and $\vec{v}$ are also solutions of the same system:

$$\begin{pmatrix} \vec{y}' = A\vec{y} \\ \& \\ \vec{y} = \vec{u} + i\vec{v} \\ \& \\ A, \vec{u}, \vec{v} \text{ real} \end{pmatrix} \Rightarrow \begin{pmatrix} \vec{u}' = A\vec{u} \\ \& \\ \vec{v}' = A\vec{v} \end{pmatrix}.$$

<u>Proof</u>: p. 257; same as that of Thm. 3.3 for DE-2 (see p. 14-6 in posted Notes).

So, $\underbrace{\vec{u} = \mathrm{Re}}_{\text{real part}}(e^{\lambda t}\vec{x})$ and $\underbrace{\vec{v} = \mathrm{Im}}_{\text{imag. part}}(e^{\lambda t}\vec{x})$

are real solutions of the lin. system.

<u>Ex. 7</u> (see also Ex. 1 in Sec. 4.6 in book)

Find a real FS of

$$\vec{y}' = \underbrace{\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}}_{A} \vec{y}.$$

<u>Sol'n</u>: 1) Find eigenvalues of A.

$$\begin{vmatrix} 0-\lambda & 1 \\ -4 & 0-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$$

2) Find eigenvectors of A.

$\underline{\lambda_1 = -2i}$ $\quad \begin{pmatrix} 0-(-2i) & 1 \\ -4 & 0-(-2i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 2i & 1 \\ -4 & 2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since the matrix on the lhs. is singular, its two eqs. must be proportional to each other, $\Rightarrow$ consider only one (say, the first):

$$2i\, x_1 + 1 \cdot x_2 = 0 \Rightarrow x_2 = -2i \cdot x_1.$$

Take $x_1 = 1 \Rightarrow x_2 = -2i$. So

$$\vec{x_1} = \begin{pmatrix} 1 \\ -2i \end{pmatrix}.$$

$\underline{\lambda_2 = 2i}$ $\quad \begin{pmatrix} -2i & 1 \\ -4 & -2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$

$-2i x_1 + x_2 = 0 \Rightarrow x_2 = 2i x_1.$

$x_1 = 1 \Rightarrow x_2 = 2i, \Rightarrow$

$$\vec{x_2} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}.$$

3) Find real solutions and verify that they form a FS.

We have $\vec{y_1} = e^{-2it} \begin{pmatrix} 1 \\ -2i \end{pmatrix}$, $\vec{y_2} = e^{2it} \begin{pmatrix} 1 \\ 2i \end{pmatrix}$.

Thm. 4.7 says we need to take Re & Im of them. Let's take $\vec{y_1}$.

$$\vec{y_1} = (\cos 2t - i \cdot \sin 2t) \Bigg[\overbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}^{\text{Real}} + \overbrace{\begin{pmatrix} 0 \\ -2i \end{pmatrix}}^{\text{Purely imaginary}} \Bigg]$$

and rewrite it as $\vec{u} + i\vec{v}$ for real $\vec{u}, \vec{v}$.

$$\vec{y}_1 = \left\{ \cos 2t \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - i \overbrace{\begin{pmatrix} 0 \\ -2i \end{pmatrix}}^{\sin 2t} \right\} +$$
$$\left\{ \cos 2t \begin{pmatrix} 0 \\ -2i \end{pmatrix} - i \sin 2t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

$$= \underbrace{\begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix}}_{\vec{u}_1} + i \underbrace{\begin{pmatrix} -\sin 2t \\ -2\cos 2t \end{pmatrix}}_{\vec{v}_1}$$

If we consider $\vec{y}_2$, we'll find $\vec{u}_2 = \vec{u}_1$, $\vec{v}_2 = -\vec{v}_1$

Thus, we have two sets : $\{\vec{u}_1, \vec{v}_1\}$ & $\{\vec{u}_1, -\vec{v}_1\}$ it is clear that they are equivalent ~~in~~ regard to what span of solutions they can represent.

Take $\{\vec{u}_1, -\vec{v}_1\} = \left\{ \begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix}, \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix} \right\}$

$$W(0) = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2 \neq 0, \Rightarrow \text{they form a FS}$$