

Lecture 27: Using LT to solve Initial Value Problems

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① Goal of this lecture

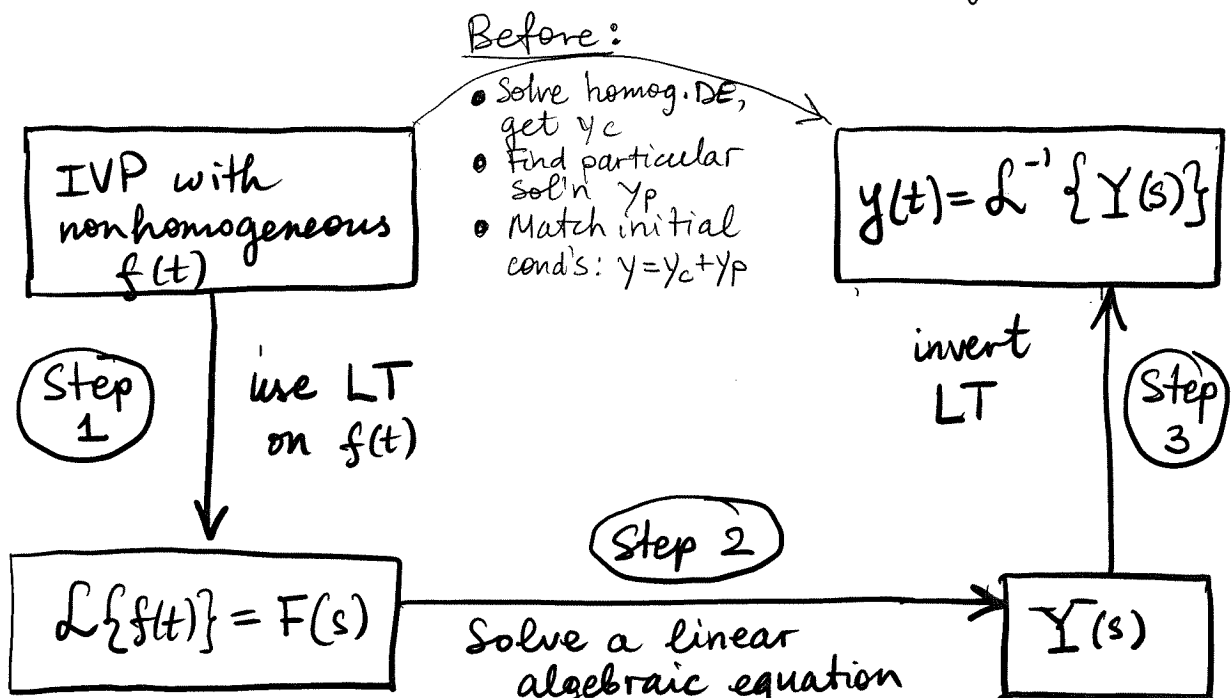
Recall the goal announced in Lec. 26:

Instead of solving an IVP using DE-solving techniques studied in earlier Chapters, we will solve it in a round-about way.

IVP to be solved:

$$\begin{cases} \text{"homogeneous DE for } y(t)\text{"} = f(t) \\ \text{Initial Conditions for } y(t_0) \end{cases}$$

← To be consistent with notations in the book, switch from "g(t)" to "f(t)" for the nonhomogeneous term.



to get: $Y(s) = \underbrace{\Phi(s)}_{\text{from homogeneous DE}} (F(s) + \underbrace{I(s)}_{\text{from Initial cond's}})$

$\underbrace{}_{\text{from } f(t)}$

So, to develop this new method of solving a nonhomogeneous DE, we will:

For step 1) Continue learning about LT;

For step 2) Learn how to find $\Phi(s)$ and $I(s)$ above;

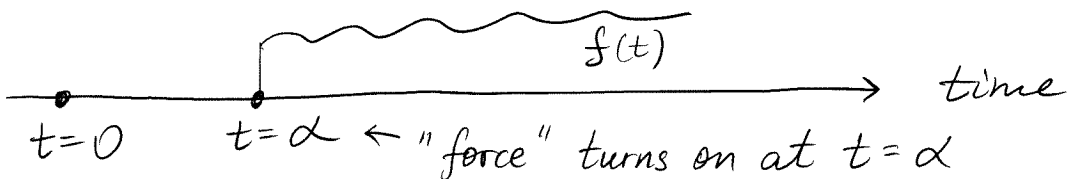
For step 3) Review how to invert LT

(we already did one example of $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ in Ex. 2 of Lec. 26.)

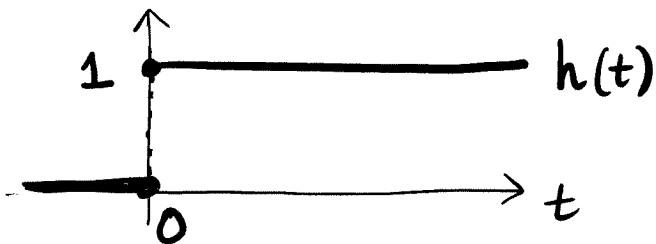
① For Step 1

1a The Heavyside step-function

In some applications, one has the nonhomogeneous terms that can begin acting after the initial time:



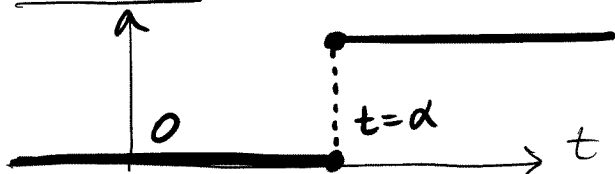
To describe this, use the Heavyside step-function:



$$h(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

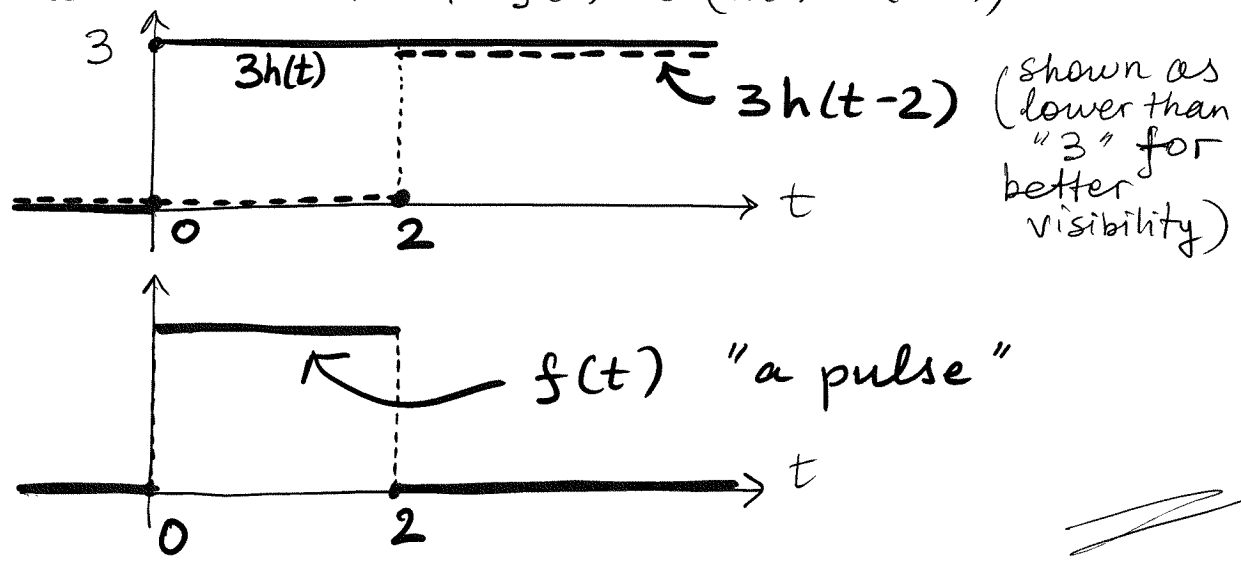
Note: The value at a single point, $t=0$, is not important.

Ex. 1(a)



← $h(t-\alpha)$

Ex. 1(b) Find $f(t) = 3(h(t) - h(t-2))$



Let us now find $\mathcal{L}\{h(t)\}$:

$$\mathcal{L}\{h(t)\} = \int_0^{\infty} e^{-st} \cdot \underset{\substack{\uparrow \\ h(t) \text{ for } t > 0}}{1} dt = \frac{e^{-st}}{-s} \Big|_0^{\infty} = \frac{e^{-\infty} - e^{-0}}{-s} = \frac{0 - 1}{-s} = \frac{1}{s}$$

See Ex. 1/lec 26 or Ex. 2/Sec. 5.1 for a more accurate treatment

Note 1: This appears to be the same answer as $\mathcal{L}\{1\}$ (see Ex. 1(a) in lec. 26 (with $a=0$) or the Table on p. 340-341 in the book). This is expected, since LT only "cares about" $t > 0$, where $h(t) = 1$.

Note 2: The real utility of $h(t)$ is to describe processes that start at $t = \alpha > 0$, and so let's find $\mathcal{L}\{h(t-\alpha)\}$.

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$$\mathcal{L}\{h(t-\alpha)\} = \int_0^{\infty} e^{-st} h(t-\alpha) dt = \int_0^{\alpha} e^{-st} \cdot 0 dt + \int_{\alpha}^{\infty} e^{-st} \cdot 1 dt = \int_{\alpha}^{\infty} e^{-st} dt =$$

$$= \int_{\alpha}^{\infty} e^{-s[(t-\alpha)+\alpha]} dt = e^{-\alpha s} \int_{\alpha}^{\infty} e^{-s(t-\alpha)} dt$$

trick to be repeatedly used below

$$\begin{array}{l} u = t - \alpha \\ t: \alpha \rightarrow \infty \\ u: 0 \rightarrow \infty \\ du = dt \end{array}$$

$$= e^{-\alpha s} \int_0^{\infty} e^{-su} du = e^{-\alpha s} \cdot \frac{1}{s}$$

\uparrow
 $\mathcal{L}\{h(u)\}$

Thus:

$$\mathcal{L}\{h(t-\alpha)\} = \frac{e^{-\alpha s}}{s}$$

1b LT of e^{at} , t^n , $\sin \omega t$, $\cos \omega t$

The first one was done in Ex. 1(a) in Lec. 26; the rest are done on pp. 330-331 in the book. (Also, $\mathcal{L}\{t^4\}$ was done in Ex. 1(b) in Lec. 26.)

They are all done via integration by parts.

See Table on pp. 340-341; you will need LT of $h(t)$, e^{at} , t^n , \sin , \cos for HW & Final Exam, but you are not required to remember any of them.

Must also see Ex. 1/Sec. 5.2 for some representative cases.

1c Shift Theorems

Under the usual conditions on $f(t)$ when all the LT's below exist (see Lec. 26, topic (4) = Thm. 5.1), one has:

Theorem 5.4 (a)

$$\mathcal{L}\{e^{\alpha t} f(t)\} = F(s-\alpha)$$

Equivalently:

$$e^{\alpha t} \underbrace{f(t)} = \mathcal{L}^{-1}\{F(s-\alpha)\} \\ = \mathcal{L}^{-1}\{F(s)\}$$

Theorem 5.4 (b)

$$\mathcal{L}\{h(t-\alpha) f(t-\alpha)\} = e^{-\alpha s} F(s)$$

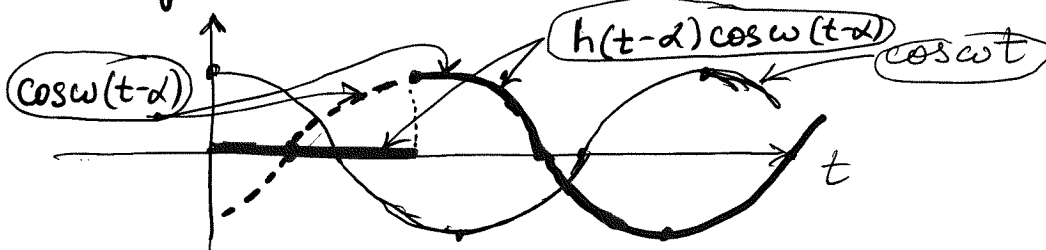
Equivalently:

$$h(t-\alpha) \underbrace{f(t-\alpha)} = \mathcal{L}^{-1}\{e^{-\alpha s} F(s)\} \\ \text{where } f(t) = \mathcal{L}^{-1}\{F(s)\}$$

See proofs of both (a) & (b) in the book (optional); the proof of (b) is analogous to the proof in Note 2 on pp. 27-3, 27-4 above.

Note 1 = Clarification about the difference between $h(t-\alpha)f(t-\alpha)$ & $f(t-\alpha)$:

- $f(t-\alpha)$ is a shifted copy of $f(t)$; in general, $f(t-\alpha) \neq 0$ for $0 < t < \alpha$.
- $h(t-\alpha)f(t-\alpha)$ is a shifted copy of $f(t)$ which begins at $t=\alpha$; $h(t-\alpha)f(t-\alpha) \equiv 0$ for $0 < t < \alpha$.



Note 2 = "do not invent your own formulas"

We know from Thm. 5.2 (Lec. 26, topic ⑤) that

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}.$$

↖ const ↗

But: $\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\}$

and similarly,

$$\mathcal{L}^{-1}\{F(s)G(s)\} \neq \mathcal{L}^{-1}\{F(s)\} \mathcal{L}^{-1}\{G(s)\}$$

E.g., from Thm. 5.4(a):

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a) \neq \mathcal{L}\{e^{at}\} \mathcal{L}\{f(t)\} (= \frac{1}{s} \cdot F(s)).$$

A more complicated (and interesting!) relation between $\mathcal{L}\{fg\}$ & $\mathcal{L}\{f\}$, $\mathcal{L}\{g\}$ does exist, but we won't have the time to learn it (Sec. 5.6, Thm. 5.7).

Note 3: The Shift Theorems allow us to expand the Table on pp. 340-341; see entries 9-12 & 15 there and also Ex. 2/Sec. 5.2.

Ex. 2(a) Find $\mathcal{L}\{h(t-3) \cos 4(t-3)\}$.

$$\mathcal{L}\{h(t-3) \cos 4(t-3)\} \underset{\substack{\uparrow \\ \text{Thm. 5.4(b)}}}{=} e^{-3s} \cdot \mathcal{L}\{\cos 4t\} \underset{\substack{\uparrow \\ \text{Table}}}{=} e^{-3s} \cdot \frac{s}{s^2 + 4^2}$$

(Recall Note 1 above: $\mathcal{L}\{h(t-3) \cos 4(t-3)\} \neq \mathcal{L}\{\cos 4(t-3)\}$!)

The latter needs to be found from $\cos 4(t-12) = \cos 4t \cos 12 + \sin 4t \sin 12$.)

Ex. 2(b) Find $\mathcal{L}^{-1} \left\{ \frac{e^{-s}}{s^2+4} \right\}$

In $e^{-\alpha s}$, $\alpha = 1$; the expression looks like Thm. 5.4(b).

So: $\mathcal{L}^{-1} \left\{ \frac{e^{-1 \cdot s}}{s^2+4} \right\} = h(t-1) f(t-1)$, where

$$f(t) = \mathcal{L}^{-1} \left\{ \frac{1}{s^2+4} \right\} \xrightarrow{\text{Table}} \mathcal{L}^{-1} \left\{ \frac{1}{2} \cdot \frac{2}{s^2+2^2} \right\} = \frac{1}{2} \sin 2t$$

So, $f(t-1) = \frac{1}{2} \sin 2(t-1)$; Answer: $h(t-1) \cdot \frac{1}{2} \sin 2(t-1)$.

Ex. 2(c) Find $\mathcal{L}^{-1} \left\{ \frac{e^{-s}}{(s-3)^2+4} \right\}$.

$$\begin{aligned} \text{This} &= h(t-1) f(t-1), \text{ where } f(t) = \mathcal{L}^{-1} \left\{ \frac{1}{(s-3)^2+4} \right\} = \\ &= e^{3t} \cdot \mathcal{L}^{-1} \left\{ \frac{1}{s^2+4} \right\} = e^{3t} \cdot \frac{1}{2} \cdot \sin 2t. \end{aligned}$$

\uparrow Thm. 5.4(a) \uparrow Ex. 2(b) $\uparrow \alpha$

Answer: $h(t-1) \cdot \frac{1}{2} e^{3(t-1)} \cdot \sin 2(t-1)$

MUST also READ Ex. 2 in textbook (Sec. 5.2).

2 For Steps 2 & 3 (see p. 27-2)

To take LT of a DE, we need:

Thm. 5.5 Assuming that all the LTs below exist:

(a) $\mathcal{L} \{ y'(t) \} = s \cdot \mathcal{L} \{ y(t) \} - y(0) \equiv s \cdot Y(s) - y(0)$

(b) $\mathcal{L} \{ y''(t) \} = s^2 Y(s) - s y(0) - y'(0)$

(will not need part (c) from the textbook).

Proof of (a): Integrate by parts

$$\begin{aligned} \mathcal{L}\{y'(t)\} &= \int_0^\infty \underbrace{e^{-st}}_g \underbrace{y'(t)}_{f'} dt = \int_a^b g f' dt = g f \Big|_a^b - \int_a^b g' f dt \\ &= e^{-st} \cdot y(t) \Big|_0^\infty - \int_0^\infty \underbrace{(-s)}_{g'} \underbrace{e^{-st}}_f \cdot y(t) dt \\ &= \underbrace{e^{-\infty}}_0 y(\infty) - \underbrace{e^{-0}}_1 y(0) + s \int_0^\infty e^{-st} y(t) dt \\ &= s Y(s) - y(0). \end{aligned}$$

Proof of (b): $\mathcal{L}\{y''(t)\} = \mathcal{L}\{(y')'\} =$
 See p. 334 (a 1-liner) \rightarrow Apply part (a) twice.

Ex. 3(a) (= simplified Ex. 3/book, so also see Ex. 3/book)

Solve the IVP: $y' - 3y = 6 \leftarrow f(t)$, $y(0) = 5$.

Sol'n: 1) $\mathcal{L}\{y' - 3y\} = \mathcal{L}\{6\} = 6 \cdot \mathcal{L}\{h(t)\}$
 $\mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = 6 \cdot \frac{1}{s}$ \leftarrow Table \uparrow See Note 1 after Ex. 1

$(sY(s) - y(0)) - 3Y(s) = \frac{6}{s}$

$(s-3)Y(s) = \frac{6}{s} + y(0) \rightarrow 5$

$Y(s) = \frac{1}{s-3} \cdot \left(\frac{6}{s} + 5 \right)$
 $\underbrace{\frac{1}{s-3}}_{\Phi(s)} \cdot \underbrace{\frac{6}{s}}_{F(s)} \cdot \underbrace{5}_{I(s)}$ \leftarrow See the diagram on p. 27-1.

2) We now need to find $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

$$\mathcal{L}^{-1}\{Y(s)\} = \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{(s-3)} \cdot \frac{6}{s}\right\}}_{\text{1st term}} + \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s-3} \cdot 5\right\}}_{\text{2nd term}}$$

1st term: Use **partial fractions** (see the Background Info sheet and HWs ## 5 & 7):

$$\frac{6}{s(s-3)} = \frac{A_1}{s} + \frac{A_2}{s-3}, \quad A_1 = -2, \quad A_2 = 2$$

$$\mathcal{L}^{-1}\left\{-\frac{2}{s} + \frac{2}{s-3}\right\} = \underbrace{-2h(t)}_{\text{from Table}} + \underbrace{2h(t)e^{3t}}_{\text{using Thm. 5.4(a) and the previous term}}$$

2nd term: as above

$$\mathcal{L}^{-1}\left\{\frac{5}{s-3}\right\} = 5h(t)e^{3t}$$

Answer: $y(t) = -2h(t) + \underbrace{2h(t)e^{3t} + 5h(t)e^{3t}}_{7h(t)e^{3t}}$

for $t > 0$, $h(t) = 1 \implies -2 + 7e^{3t} \quad (t > 0)$.

Ex. 3(b) Same, but for $f(t) = \begin{cases} 0, & t < 2 \\ 6, & t \geq 2 \end{cases} = 6h(t-2)$.

Sol'n:

1) Proceed as in 1) of Ex. 3(a); the difference will be only in $\mathcal{L}\{f(t)\}$:

$$\mathcal{L}\{f(t)\} \stackrel{\uparrow}{=} 6 \cdot \frac{e^{-2s}}{s} \quad \leftarrow \text{came from } h(t-2)$$

Table (#14)

or Table (#1) + Thm. 5.4(b)

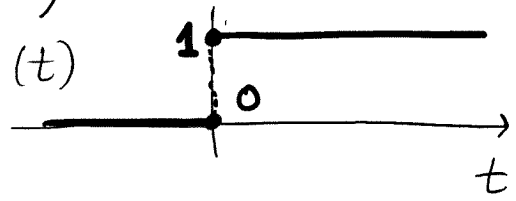
$$2) Y(s) = \frac{1}{s-3} \left(6 \frac{e^{-2s}}{s} + 5 \right)$$

$$y(t) = \mathcal{L}^{-1} \left\{ \underbrace{e^{-2s}}_{\substack{\text{use} \\ \text{Thm. 5.4(b)}}} \cdot \underbrace{\frac{6}{s(s-3)}}_{\substack{\text{from Ex. 3(a)}}} + \underbrace{\frac{5}{s-3}}_{\substack{\text{same as in} \\ \text{Ex. 3(a)}}} \right\}$$

$$h(t-2) \left(-2h(t-2) + 2h(t-2)e^{3(t-2)} \right)$$

Now use the fact: $(h(t))^2 = h(t)$

\Rightarrow



Answer: $y(t) = -2h(t-2)(1 - e^{3(t-2)}) + 5e^{3t}h(t).$

Ex. 4 Solve the IVP

$$y'' - y = e^{-2t}, \quad y(0) = y_0, \quad y'(0) = y_0'$$

Sol'n:

$$1) \mathcal{L}\{y'' - y\} = \mathcal{L}\{e^{-2t}\}$$

\downarrow Thm. 5.5 (b) Table

$$(s^2 Y(s) - s y_0 - y_0') - Y(s) = \frac{1}{s+2}$$

$$(s^2 - 1) Y(s) = \frac{1}{s+2} + s y_0 + y_0'$$

$$Y(s) = \underbrace{\frac{1}{s^2 - 1}}_{\Phi(s)} \cdot \left(\underbrace{\frac{1}{s+2}}_{F(s)} + \underbrace{s y_0 + y_0'}_{I(s)} \right)$$

See the diagram on p. 27-1

$$= \frac{1}{(s^2 - 1)(s+2)} + y_0 \frac{s}{s^2 - 1} + y_0' \frac{1}{s^2 - 1}$$

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Each of these terms is done by partial fraction expansion:

$$Y(s) =$$

$$\left(\frac{1/6}{s-1} - \frac{1/2}{s+1} + \frac{1/3}{s+2} \right) + y_0 \left(\frac{1/2}{s-1} + \frac{1/2}{s+1} \right) + y_0' \left(\frac{1/2}{s-1} - \frac{1/2}{s+1} \right)$$

2) Use inverse LT (again, use the Table):

$$y(t) = \left(\frac{1}{6} e^t - \frac{1}{2} e^{-t} + \frac{1}{3} e^{-2t} \right) + \frac{y_0}{2} (e^t + e^{-t}) + \frac{y_0'}{2} (e^t - e^{-t}).$$

