Lecture 3 General properties of solutions of linear DEs

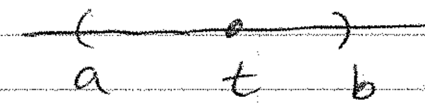
Notations:

old — don't need to state at beginning of Lecture

$$\begin{cases} y' + py = 0 & (H) \leftarrow \text{homogeneous DE} \\ y' + py = g & (NH) \leftarrow \text{non-homogeneous DE} \end{cases}$$

NEW "$t \in (a,b)$" means "$a < t < b$"

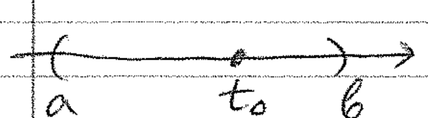
$\in$ "belongs to" or "is in".



① Existence and uniquenes of solution

Thm. 2.1 (p. 17)

Let $p(t)$, $g(t)$ be continuous for all $t \in (a,b)$. Let also $t_0 \in (a,b)$. Then the IVP



$$\begin{cases} y' + p(t)\,y = g(t) \\ y(t_0) = y_0 \end{cases} \qquad (1)$$

has a unique sol'n for all $t \in (a,b)$.

Proof: 1) Existence ("has")

This follow from formula (9) of Lec. 2, which presented an explicit solution of IVP (1):

$$y(t) = e^{-P(t)}\left(\int_{t_0}^{t} e^{P(t_1)} g(t_1)\,dt_1 + y_0\right) \quad (2a)$$

$$P(t) = \int_{t_0}^{t} p(t_2)\,dt_2 . \quad (2b)$$

Since $p(t)$ is continuous on (a,b), then so are $P(t)$ and $e^{\pm P(t)}$.
Similarly, since $g(t)$ is continuous, then so is

$$\int_{t_0}^{t} e^{P(t_1)} g(t_1)\,dt_1 ,$$ i.e. it exists. ✓

2) <u>Uniqueness</u>

The method of proof used below is commonly used in many other theorems.

<u>Proof</u> (by contradiction):

Suppose $y_1(t)$ & $y_2(t)$ are <u>two</u> sol'ns of IVP (1):

$$\begin{cases} y_1' + p y_1 = g \\ y_2' + p y_2 = g \end{cases}, \quad \begin{cases} y_1(t_0) = y_0 \\ y_2(t_0) = y_0 . \end{cases}$$

Subtract the 2nd eqs from the 1st and denote

$$W = y_1 - y_2 :$$

$$(y_1 - y_2)' + p(y_1 - y_2) = \underbrace{\cancel{g - g}}_{0}, \quad y_1(t_0) - y_2(t_0) = \underbrace{\cancel{y_0 - y_0}}_{0}$$

$$\begin{cases} W' + pW = 0 \\ W(t_0) = 0 \end{cases} \leftarrow \text{Homogeneous DE with } \underline{\text{zero}} \text{ IC.}$$

In Ex. 4 of Lec. 2 we showed that the only solution of this IVP is

$$W(t) = 0 \text{ for all } t, \Rightarrow$$

$$y_1(t) - y_2(t) = 0 \Rightarrow y_1(t) = y_2(t) \text{ for all } t.$$

1/27/16

Q: What if $p(t)$ and/or $g(t)$ are not continuous on (a,b)?
Does it mean that $y(t)$ does not exist?

A: It depends; see the next Example.

Ex. 1 (a) The IVP $y' = \frac{1}{t} y$, $y(-2) = 3$

$-p(t)$ not continuous on $(-2,1)$

has the solution $y = -\frac{3}{2} t$ (Similar to Ex. 6 of Lec. 2; also can verify by substitution).

Thus, here $y(t)$ exists and is continuous.

(b) The IVP $y' = - \frac{1}{t} y$, $y(-2) = 3$

difference from (a)

has the solution $y = -\frac{6}{t}$, which is not cont.

Moral: No general conclusion about the existence of solution can be made if $p(t)$ & $g(t)$ are not continuous.

(2) Superposition principle (not IVP)

The goal of solving a DE is to find as many of its solutions as possible.
(This becomes especially relevant and challenging for higher-order DEs, which contain y'', y''', etc.)

Hence, Q: If we know some two solutions of a DE, can we use them to construct a third solution?

A: Affirmative for linear DEs.

Thm. (Superposition principle).

$\boxed{\text{(a)}}$ Let $y_1(t)$, $y_2(t)$ be two solutions of the homogeneous DE

$$y' + py = 0 \qquad (H)$$

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Then $y_3 = c_1 y_1 + c_2 y_2$, where $c_{1,2} = \text{const}$, is another solution of (H).

Proof (similar to Proof of Uniqueness of Thm. 2.1):

$$\left.\begin{aligned} c_1 \cdot (y_1' + py_1 = 0) \\ c_2 \cdot (y_2' + py_2 = 0) \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 y_1' + c_1 p y_1 = 0 \\ c_2 y_2' + c_2 p y_2 = 0 \end{aligned}$$

$$\Rightarrow + \begin{aligned} (c_1 y_1)' + p(c_1 y_1) = 0 \\ (c_2 y_2)' + p(c_2 y_2) = 0 \end{aligned}$$

$$\underbrace{(c_1 y_1 + c_2 y_2)}_{y_3}{}' + p\underbrace{(c_1 y_1 + c_2 y_2)}_{y_3} = 0 \qquad \checkmark$$

(b) Let y_1 satisfy (H) and y_2 satisfy

$$y_2' + p y_2 = g \qquad \text{(NH)}$$

non-homogeneous

Then $y_3 = c y_1 + y_2$, where c = any const, is also a solution of (NH).

Proof: At Home, similar to part (a).

Corollary Suppose one knows one sol'n, $y_p(t)$ of (NH). To find the general sol'n of (NH), do:

particular sol'n

1) Solve (H): $y_h' + p y_h = 0$ and find any y_h.

2) General sol'n of NH:

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$$\boxed{y = c y_h + y_p, \quad c = \text{const}} \qquad (3)$$

Ex. 2 By inspection, it is easy to see that a sol'n of

$$y' = y - t + 1$$

is $y_p = t$. Find the general sol'n.

Sol'n: Find y_h s.t. $y_h' = y_h$.
By Lec. 2, $y_h = e^t$.

Then, by Corollary, the general sol'n is

$$y = c e^t + t.$$

<u>Note 1</u> If y_1, y_2 are two solutions of (NH), then $(y_1 + y_2)$ is <u>not</u> a solution of (NH).

<u>Proof</u> — at home.

<u>Note 2</u> Superposition Principle (neither (a) nor (b)) for <u>nonlinear</u> DEs is <u>not</u> valid.

(This is why <u>linear</u> DEs are so important. In contrast, for nonlinear DEs, knowing two particular solutions gives <u>no further information</u> about the general sol'n.)

<u>Ex. 3</u> Consider $y' = y^2 - t^2 + 1$. (4)

By inspection, $y_p = t$ is a particular sol'n.
Also, $y_h = -1/t$ solves $y_h' = y_h^2$
(will show this later).
Show that $y = t - c/t$ is <u>not</u> a sol'n of (4) unless $c = 0$.

<u>Sol'n:</u> <u>lhs:</u> $y' = 1 + \frac{c}{t^2}$

<u>rhs</u>: $(t - \frac{c}{t})^2 - t^2 + 1 = \cancel{t^2} - 2t \cdot \frac{c}{t} + \frac{c^2}{t^2} - \cancel{t^2} + 1$

So: $\cancel{1} + \cancel{\frac{c}{t^2}} \stackrel{?}{=} -2c + \cancel{\frac{c^2}{t^2}} + \cancel{1}$

not equal for $c \neq 0$.

<u>HW</u>: <u>Sec. 2.1</u> # 11(a), 13(b), 16; <u>Sec. 2.2</u> # 33, 31.
<u>WP1</u>: Prove part (b) of Superposition Principle
<u>WP2</u>: Similarly to WP1, prove Note 1 on p. 3-6.
<u>WP3</u>: 1) Guess a particular sol'n of the DE of #36 of Sec. 2.2. 2) Find the general sol'n. 3) Solve IVP of #36.