Lecture 5: Solution of separable DEs.

(Sec. 2.6)

Def: A DE of the form

$$\frac{dy}{dt} = -\frac{m(t)}{n(y)}$$

are called called separable. The "−" on the r.h.s. is only to conform to the notations of the book.

To solve a separable DE:

$$\frac{dy}{dt} = -\frac{m(t)}{n(y)} \Rightarrow$$

$$n(y)\,dy = -m(t)\,dt$$

$$\int n(y)\,dy = -\int m(t)\,dt \Rightarrow$$

obtain an algebraic eqn. relating y and t.

Ex. 1 Linear separable DEs

$$\frac{dy}{dt} = -p(t)\cdot y \Rightarrow \frac{dy}{y} = -p(t)dt \Rightarrow$$

$$\ln|y| = -P(t) + C_1 \Rightarrow |y| = e^{-P(t)+C_1} = e^{-P(t)}\cdot \underbrace{e^{C_1}}_{C_2 > 0}$$

Thus, we have proved Eq. (4) of Lec. 2:

$$|y(t)| = C_2 e^{-P(t)} \Rightarrow y = \underbrace{(\pm C_2)}_{C,\text{ either positive or negative, or } 0}\cdot e^{-P(t)}$$

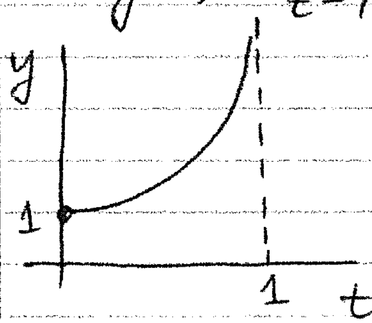
Ex. 2(a) $\quad y' = y^2, \quad y(0) = 1$

Sol'n: 1) $\dfrac{dy}{y^2} = 1 \cdot dt \Rightarrow -\dfrac{1}{y} = t + C$

$\Rightarrow y = \dfrac{-1}{t+C}$.

2) $y(0) = 1 \Rightarrow \quad 1 = \dfrac{-1}{0+C} \Rightarrow C = -1 \Rightarrow$

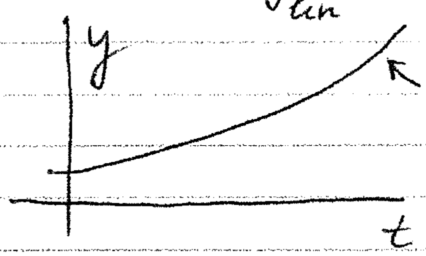
$y(t) = \dfrac{-1}{t-1} = \dfrac{1}{1-t}$.



Note that the solution __blows up__ at $t=1$, even though there are no discontinuous coeff's in the DE at all!

This should be contrasted with the solution of a linear equation, say,

$$y_{lin} = e^{(-p_0)t}, \quad (-p_0) > 0 \quad :$$



$e^{(-p_0)t}$

$(-p_0) > 0$

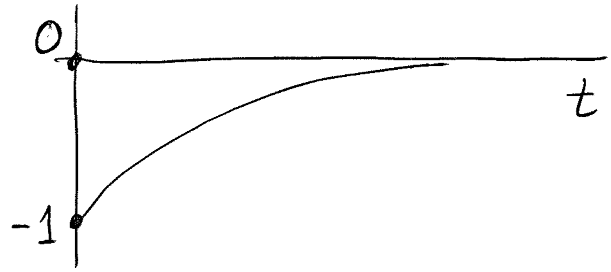
It grows exponentially, but exists for all t.

Ex. 2(b) $\quad y' = y^2, \quad y(0) = -1$

Sol'n: 1) $y = \dfrac{-1}{t+C}$ (same as in 2(a));

2) $-1 = \dfrac{-1}{0+C} \Rightarrow C = 1 \Rightarrow y = \dfrac{-1}{1+t}$.

Note that this solution decays to 0:



<u>Ex. 2(c)</u> $y' = -y^2$; $y(0) = 1$

<u>Sol'n:</u> 1) $\frac{1}{y} = t + C$ (similarly to Ex. 2(a))

2) Use the initial condition to find C:

$$\frac{1}{1} = 0 + C \Rightarrow C = 1 \Rightarrow y = \frac{1}{t+1}.$$

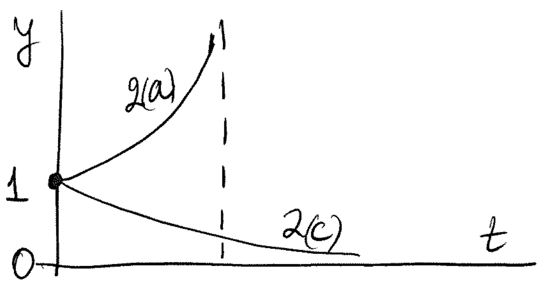
Again, this solution decays to 0.

<u>Ex. 2(d)</u> $y' = -y^2$, $y(0) = -1$

Similarly to Ex. 2(a), the solution blows up at $t = 1$.

<u>General Note #1</u> <u>Qualitative behavior of the solution</u>

Compare Ex. 2(a) & 2(c).



<u>2(a):</u> $y' = y^2 > 0 \Rightarrow y \nearrow$

<u>2(c):</u> $y' = -y^2 < 0 \Rightarrow y \searrow$

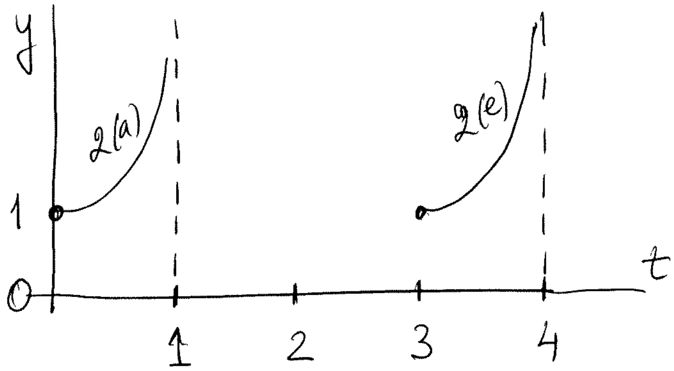
(Of course, without actually solving the DE, we cannot predict whether a blow-up will occur.)

<u>Ex. 2(e):</u> $y' = y^2$, $y(3) = 1$

1) $y = \frac{-1}{t+C}$ (same as in Ex. 2(a))

2) To find C: $1 = \frac{-1}{3+C} \Rightarrow C+3 = -1 \Rightarrow C = -1-3$

$$y = \frac{-1}{t \underbrace{-1-3}_{C}} = \frac{-1}{(t-3)-1} = \frac{1}{1-(t-3)} \quad \left(\text{also} = \frac{1}{4-t}\right).$$



Note that this is just a translated curve of 2(a).

This is the general case for autonomous DEs.

General note # 2

Consider two IVPs with the same autonomous DE

(I) $y' = f(y)$, $y(0) = y_0$; let its sol'n be $y_I(t)$.

(II) $y' = f(y)$, $y(t_0) = y_0$; let its sol'n be $y_{II}(t)$.

Claim:

$$y_{II}(t) = \underbrace{y_I(t-t_0)}_{\text{"}y_I(t)\text{ shifted to the right by }t_0\text{"}}$$

Indeed: Rewrite (II) as:

- $\underbrace{\frac{dy}{d(t-t_0)}}_{\text{since } dt = d(t-t_0),\text{ as } dt_0 = 0} = f(y)$, $\underbrace{y(t=t_0)}_{y(t-t_0=0)} = y_0$
- Denote $(t-t_0) = s$, a new variable

- Rewrite the IVP – II again:

$$\frac{dy}{ds} = f(y), \quad \underbrace{y(s=0)}_{y(0)} = y_0$$

But this is the same as IVP –(I), with t being replaced by $s = t - t_0$; hence indeed

$$y_{II}(t) = y_I(s) = y_I(t - t_0).$$

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<u>Ex. 2 (f)</u> $\quad y' = y^2, \quad y(0) = 0$

1) $y = \dfrac{-1}{t+C}$

2) Find C: $\quad 0 = \dfrac{-1}{0+C} \Rightarrow C = \infty, \Rightarrow$

$$y = \frac{-1}{t+\infty} = 0 \Rightarrow y = 0.$$

This could be guessed by simple inspection: $y = 0$ satisfies: $y' = 0$ and $y^2 = 0$ and $y(0) = 0$.

<u>Moral from Ex. 2 (f)</u>: For some nonlinear DEs with some special initial conditions, the solution can be found by simple inspection.

<u>General Note # 3</u> Sometimes the algebraic eq. that can be obtained from

$$\int n(y)dy = -\int m(t)dt$$

<u>cannot</u> be solved for y. Then leave that algebraic equation in the form:

$$h(t,y) = C$$

↑ some nonlinear function.

Its solution $y(t)$ can be found (and plotted) on a computer. (See the result of Ex. 2,3 of Sec. 2.6/book, but do not use their method, as it is too complicated.)

General Note # 4

Any autonomous DE $\frac{dy}{dt} = f(y)$

is separable : $\frac{dy}{f(y)} = dt$

$$\left(n(y) \equiv \frac{1}{f(y)}, \quad m(t) \equiv -1 \right)$$