Lecture 9   Euler's method

Goal: Solve $\dfrac{dy}{dt} = f(t, y)$ approximately

when it cannot be solved exactly.

① Idea: Approximate dy/dt:

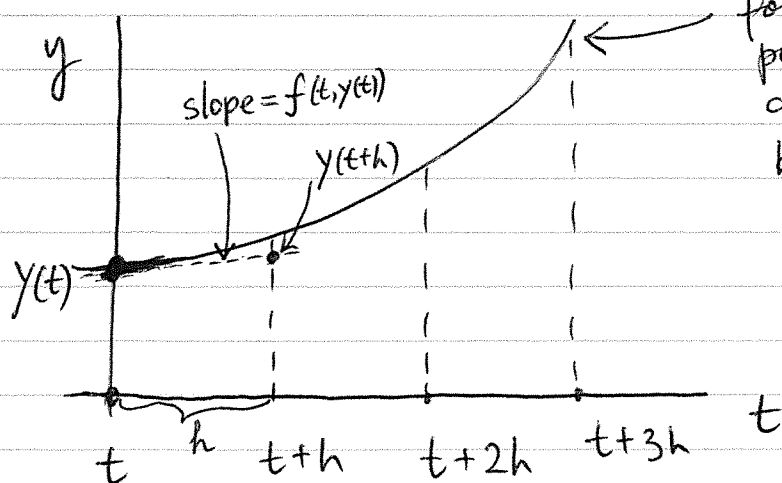$$\frac{dy}{dt} \approx \frac{y(t+h) - y(t)}{h} \quad \text{for small } h.$$

So, replace the DE with:

$$\frac{y(t+h) - y(t)}{h} = f(t, y(t))$$

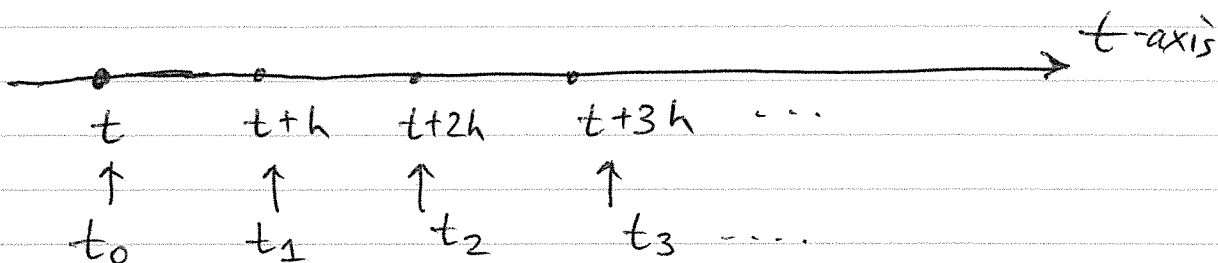and solve for the "new" value $y(t+h)$ using the known "old" value $y(t)$:

$$\boxed{y(t+h) = y(t) + h \cdot f(t, y(t))}$$ Euler's Method.

② Graphic representation

slope = $f(t, y(t))$

$y(t+h)$

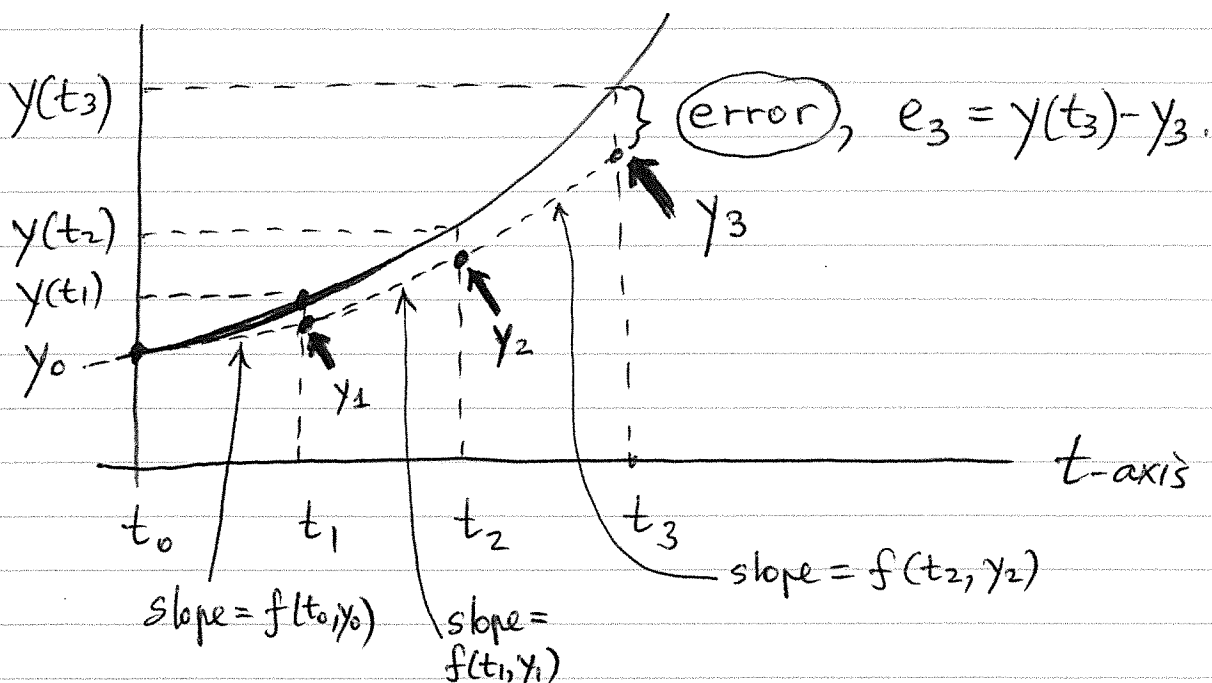$y(t)$

t h $t+h$ $t+2h$ $t+3h$ t

for this illustration purpose, let's assume that we know the sol'n $y(t)$

Let's introduce a more convenient notation:

$$t_k = t_0 + k \cdot h, \quad k = 0, 1, 2, \ldots$$

Then we can redraw the previous figure:

$$\boxed{error}, \quad e_3 = y(t_3) - y_3.$$

slope $= f(t_0, y_0)$

slope $= f(t_1, y_1)$

slope $= f(t_2, y_2)$

Another notation: write Euler's method as

$$\boxed{y_{k+1} = y_k + h \cdot f(t_k, y_k)}$$

$y_k = \boxed{approximate}$ solution at $t = t_k$.

$y(t_k) = \boxed{exact}$ (and usually unknown) solution at $t = t_k$

Ex.1 Compute y_1, y_2, y_3 for

$$y' = 0.5y + t, \qquad y(t_0) = 3; \quad t_0 = 0;$$

while using $h = 0.1$.

Sol'n: $\qquad y_{k+1} = y_k + h \cdot f(t_k, y_k)$

$\qquad\qquad = 3 \Rightarrow$ *k = 2.* ← **Stop at $k = 2$** **(for y_3)**

Decide on the k where you need to stop.

Thus need to do the calculations for $k = 0, 1, 2$.

$\underline{k = 0}$ $\qquad y_{\underset{1}{0+1}} = y_0 + h \cdot f(t_0, y_0)$

$$= 3 + 0.1 \cdot (0.5 \cdot \overset{3}{y_0} + \overset{0}{t_0}) \qquad \swarrow y_1$$

$$= 3 + 0.1 \cdot (0.5 \cdot 3 + 0) = \boxed{3.15}$$

$\underline{k = 1}$

$$y_{\underset{2}{1+1}} = y_1 + h \cdot f(t_1, y_1)$$

$$= 3.15 + 0.1 \cdot (0.5 \cdot \overset{3.15}{y_1} + \overset{0.1}{t_1}) \qquad \swarrow y_2$$

$$= 3.15 + 0.1 \cdot 1.675 = \boxed{3.3175}$$

$\underline{k = 2}$ $\quad y_{\underset{3}{2+1}} = y_2 + h \cdot f(t_2, y_2)$

$$= 3.3175 + 0.1 \cdot (0.5 \cdot \overset{3.3175}{y_2} + \overset{0.2}{t_2}) \qquad \swarrow y_3$$

$$= 3.3175 + 0.1 \cdot 1.85875 = \boxed{3.503375}$$

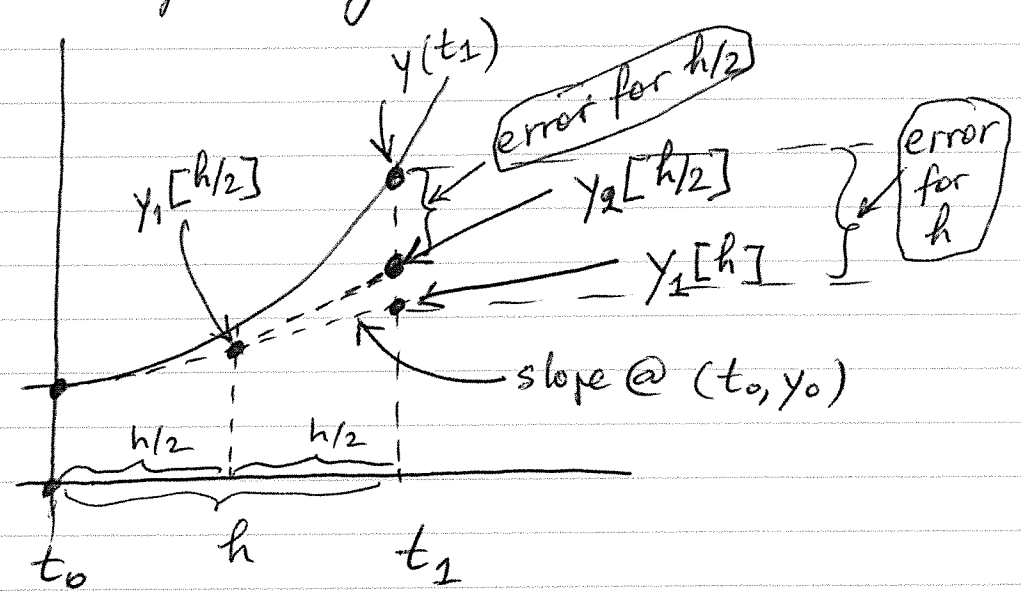Exact sol'n (Lec. 2): $y(\underset{0.3}{t_3}) = 3.53284$.

The error at $t_3 = 0.3$ is

$$|y(t_3) - y_3| = |3.53284 - 3.503375| \approx 0.0295,$$

or $\lesssim 0.9\%$.

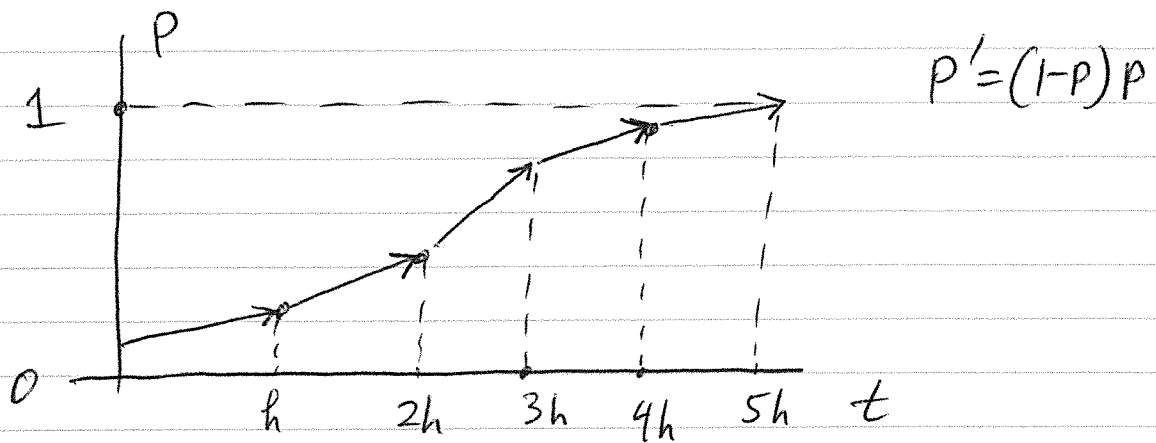Note: The error will decrease if we take a smaller h (expectedly)!

Why? Graphically:

We see that $y_2\left[\tfrac{h}{2}\right]$ is a better approximation to $y(t_1)$ than $y_1[h]$.

Mnemonically: The more accurate solution $y_{2k}\left[\tfrac{h}{2}\right]$ is always found between the exact sol'n $y(t_k)$ and the less accurate sol'n $y_k[h]$.

③ Relation to direction field:

$$P' = (1-P)P$$

The approximate solution found by Euler's method follows arrows of the direction field.

HW: Sec. 2.10 ## 3, 5.
Note: 1) Do _not_ compute y_3 & e_3.
for both

2) In addition to the assigned calculation, repeat it for $h = 0.05$ and compute y_1, y_2, y_3, y_4 for this new h, as well as e_1, e_2, e_3, e_4.

Verify that $e_4 [h=0.05] \approx \frac{1}{2} e_2 [h=0.1]$.

(In general, $e_k = y(t_k) - y_k$ for a given h.)