

Sec. 14.1 Functions of several variables

① Motivation and basic definitions

Most quantities in nature depend on more than 1 variable.

E.g., fuel efficiency of your car depends on such factors as:

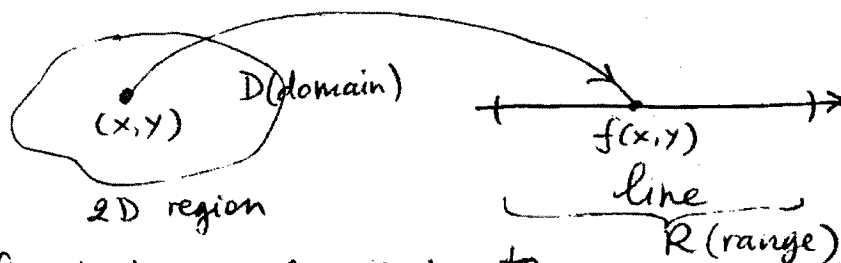
- tire pressure;
- weight load;
- velocity of wind relative to velocity of car;
- elevation angle of road.

So:

$$\underset{\text{fuel efficiency}}{E} = f(P, W, A, \dots).$$

See also Exs. 2, 3 in book.

Def. of a function of 2 variables
(see book p. 855 for a rigorous form)



$f(x, y)$ is a rule that ~~for~~^{to} each point (x, y) in a 2D domain D puts in correspondence a point on a ~~2D~~ 1D line within some range R .

skip, refer
to book.

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Ex. 1 (see also Ex. 14 in book)

Given $f(x,y) = x^2 + \sqrt{x+y^2}$, find:

- 1) $f(-2,3)$; 2) domain of f ; 3) range of f .

Sol'n: 1) $f(-2,3) = (-2)^2 + \sqrt{-2+3^2} = 4 + \sqrt{7}$.

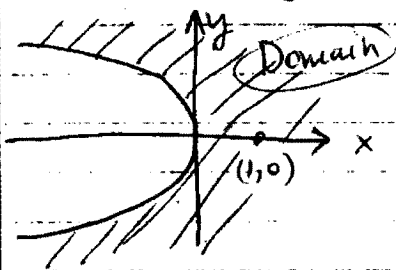
- 2) Domain of f is where each term in it is defined.

x^2 is defined for all x, y .

$\sqrt{x+y^2}$ is defined where $x+y^2 \geq 0$.

The boundary of the domain is:

$$x+y^2=0, \quad x=-y^2.$$



We need $x+y^2 \geq 0$,
so pick any pt., say
 $(x=1, y=0)$: $1+0^2=1 \geq 0$.

So $x+y^2 \geq 0$ is on that side
of curve $x+y^2=0$ where $(1,0)$ is.

- 3) Range is what values f can take.

$$f = \underbrace{x^2}_{\geq 0} + \underbrace{\sqrt{x+y^2}}_{\geq 0} \geq 0.$$

So the range of f is $[0, \infty)$.

② Graphs of $f(x,y)$.

This is just the surface $z=f(x,y)$.

(see also Ex. 5, 6, 8 in book)
Ex. 1 Sketch the graph of $f(x, y)$:

(a) $f(x, y) = x - 2y + 3$.

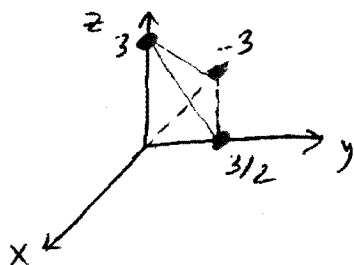
(b) $f(x, y) = 1 - x^2$

(c) $f(x, y) = x^2 + 2y^2 - 3$

Sol'n: (a) $z = x - 2y + 3$

$$x - 2y - z + 3 = 0$$

This is a plane $\perp$ to $\langle 1, -2, -1 \rangle$.



$$z = 0 : x - 2y + 3 = 0$$

$$\underline{x=0} \Rightarrow -2y + 3 = 0 \Rightarrow y = 3/2$$

$$\underline{y=0} \Rightarrow x + 3 = 0 \Rightarrow x = -3$$

$$\underline{x=y=0} \Rightarrow z = 3$$

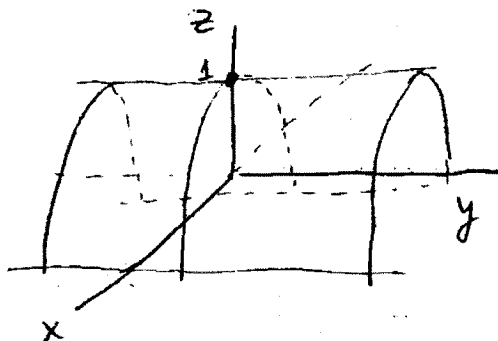
This plane has the intercepts as above.

In general, $f(x, y) = ax + by + c$ for any a, b, c is the eq. of a plane. It is called a linear function of (x, y) (in analogy with $y = Ax + B$ being a linear function of x).

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(b) $z = 1 - x^2$

This is a parabolic cylinder with axis along y -axis (Sec. 12.6).

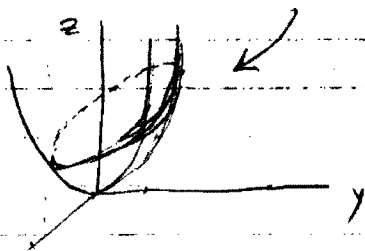


(c) $z = x^2 + 2y^2 - 3$

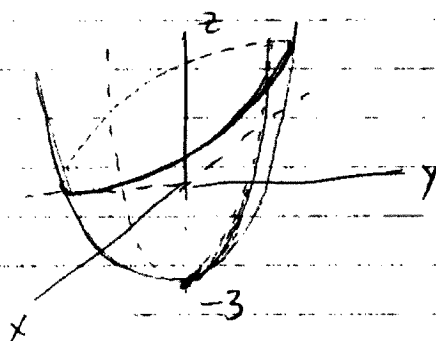
$z = x^2 + 2y^2$ is an elliptical paraboloid.

(Sec. 12.6)

$z = \frac{x^2}{1} + \frac{y^2}{(1/2)}$



$z = x^2 + 2y^2 - 3$ is this paraboloid lowered by 3 units:



③ Level curves (contour plots)

If you cannot easily sketch $z = f(x,y)$, try to consider its traces in planes $z = K$ ($= \text{const}$).
(Recall Lab 3.)

So, each trace is a curve (called level curve):

$$z = K, \quad K = f(x,y).$$

The last equation is that of a 2D curve, which may be easier to sketch.

Ex. 2 Sketch:

(a) $(x+y)^2$ and

(b) $\sin(x+y)$

by considering their level curves.

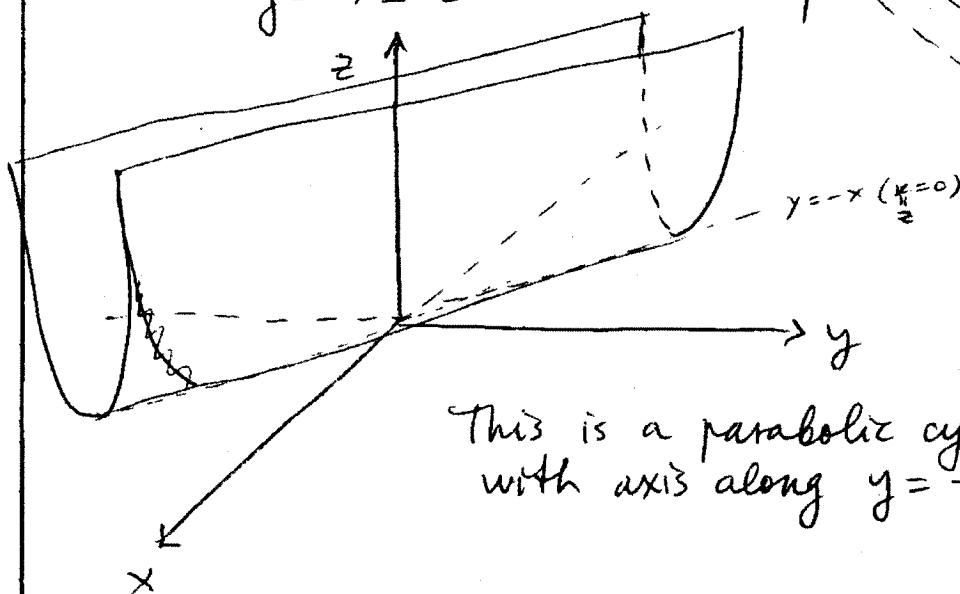
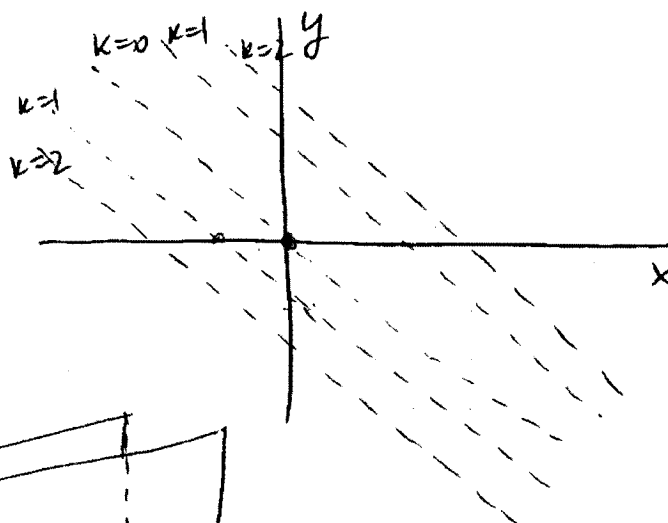
(a) $(x+y)^2 = k$.

This can hold only for $k \geq 0$,
so the surface $z = (x+y)^2$ lies on
or above the xy -plane ($z \geq 0$).

$k=0$: $(x+y)^2 = 0 \Rightarrow x+y=0 \Rightarrow y=-x$

$k=1$ $(x+y)^2 = 1$
 $x+y = \pm 1$
 $y = -x \pm 1$

$k=2$ $(x+y)^2 = 2$
 $x+y = \pm \sqrt{2}$
 $y = -x \pm \sqrt{2}$



This is a parabolic cylinder
with axis along $y = -x$.

(b) $\sin(x+y) = k$

This can hold only for $-1 \leq k \leq 1$,
so the surface $z = \sin(x+y)$ lies between planes
 $z = -1$ and $z = 1$.

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$$k=0 \quad \sin(x+y)=0 \Rightarrow x+y=0, \pm\pi, \pm 2\pi, \text{ etc.}$$

$$y=-x, -x\pm\pi, -x\pm 2\pi, \dots$$

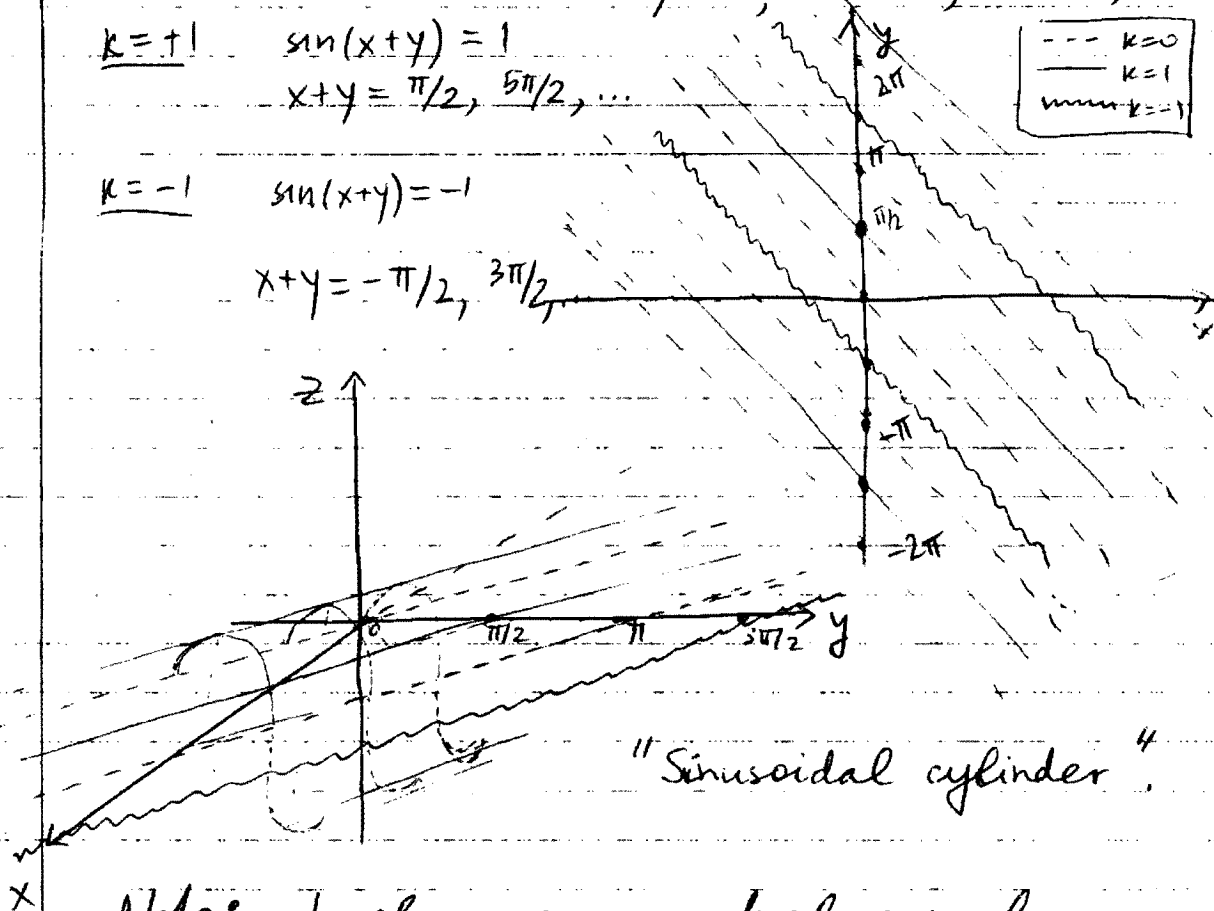
$$k=+1 \quad \sin(x+y)=1$$

$$x+y=\pi/2, 5\pi/2, \dots$$

$$k=-1 \quad \sin(x+y)=-1$$

$$x+y=-\pi/2, 3\pi/2, \dots$$

---	$k=0$
—	$k=1$
---	$k=-1$



Note: Level curves cannot always be easily drawn. E.g.: $\sin x + \sin y = k$. However, drawing level curves is usually a good first step to try.

Notation: A collection of level curves for various k is called a contour plot (or contour map).

equidistant

A well-known example of a contour plot is a topographic map, where level curves denote regions of equal height.

Also: temperature or atmospheric pressure maps, etc.

Ex. 3 (a) Make a contour plot of a paraboloid
 $z = x^2 + y^2$.

(b) Based on this plot, comment how the steepness of this surface changes as the point on the surface moves away from $(0,0,0)$.

Sol'n: (a) We need to draw several plots of
 $x^2 + y^2 = K$. (obviously, $K \geq 0$).

$K=0$ $x^2 + y^2 = 0 \rightarrow (0,0)$.

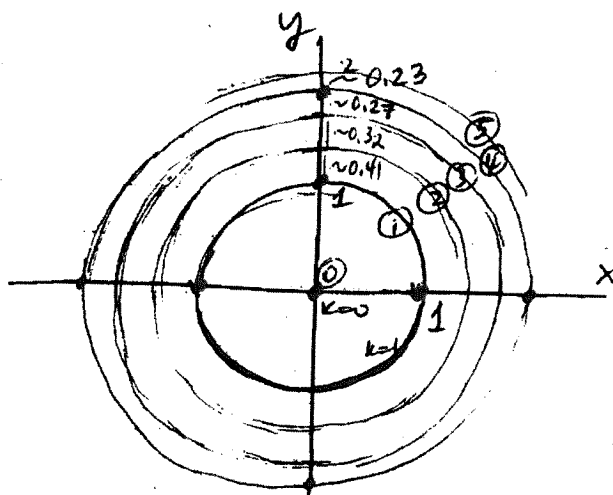
$K=1$ $x^2 + y^2 = 1$
 circle, radius = 1

$K=2$ $x^2 + y^2 = (\sqrt{2})^2$
 radius = $\sqrt{2}$

$K=3$ $x^2 + y^2 = 3 = (\sqrt{3})^2$
 rad. = $\sqrt{3}$

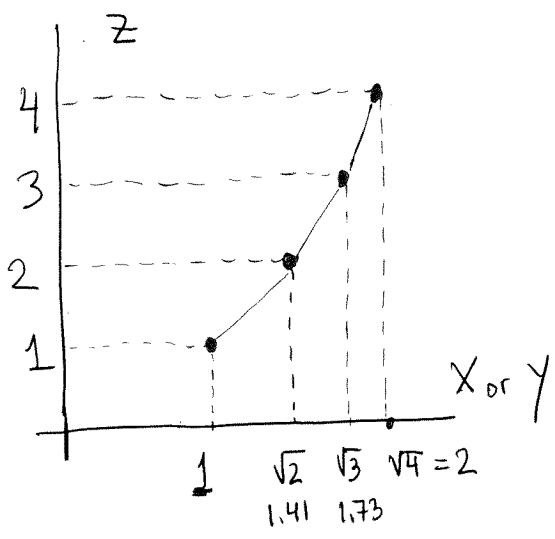
$K=4$ $x^2 + y^2 = 4 = 2^2$
 rad. = 2

$K=5$ $x^2 + y^2 = (\sqrt{5})^2$
 rad = $\sqrt{5} \approx 2.23$



As $K \uparrow$,
 the circles
 become closer
 and closer.

(b) As we go from K -th level curve to the $(K+1)$ -th, we need to make smaller and smaller steps in the xy -plane, as $K \uparrow$.



Slope = $\frac{\Delta z}{\Delta x}$ = 1 by assumption
← decreases

⇒ slope ↑ as x ↑ in this case.

Moral:

- Level curves crammed close together indicate a steep surface;
- Level curves spaced far apart indicate a surface with gentle slope.

④ Functions of 3 variables



Examples:

"rho" → $\rho(x, y, z)$ ← density of a 3D object
 $T(x, y, z)$ ← temperature of a 3D object

Level curves in 2D become level surfaces in 3D,

Example: $T(x, y, z) = K$ defines a surface of equal temperature inside a 3D object.

See also Ex. 15 in textbook.