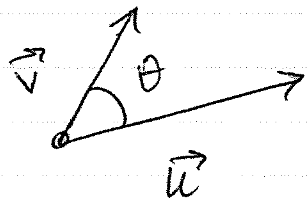
Review of Dot product (Sec. 12.3)

① Definition / formula



$$\vec{u} \bullet \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$\langle u_1, u_2, u_3 \rangle$ $\quad$ $\langle v_1, v_2, v_3 \rangle$

MEMORIZE BOTH

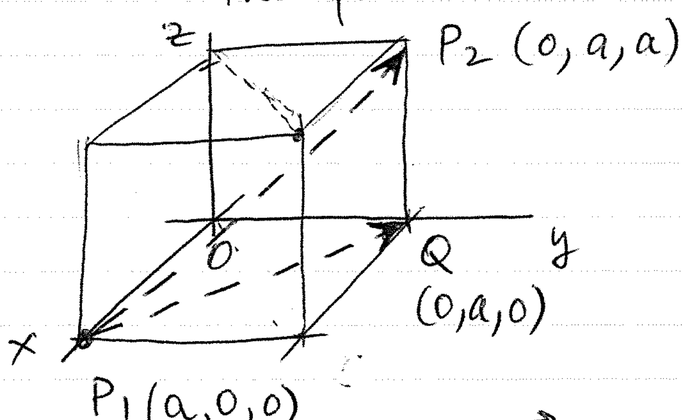
$$\vec{u} \bullet \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos\theta \qquad (\bigstar)$$

Note 1: the dot product is a scalar, not a vector.

Note 2: $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$

Formula ($\bigstar$) can be used to find the angle between vectors.

Ex. 1 Find the angle between a diagonal of a cube and its projection onto one of the faces.



Sol'n:

1) State formula:

$$\cos\theta = \frac{\overrightarrow{P_1P_2} \bullet \overrightarrow{P_1Q}}{|\overrightarrow{P_1P_2}| \cdot |\overrightarrow{P_1Q}|}$$

2) $\overrightarrow{P_1P_2} = \langle 0-a,\ a-0,\ a-0 \rangle$
$= \langle -a, a, a \rangle.$

$\overrightarrow{P_1Q} = \langle 0-a,\ a-0,\ 0-0 \rangle = \langle -a, a, 0 \rangle.$

3) $|\overrightarrow{P_1P_2}| = \sqrt{(-a)^2+a^2+a^2} = \sqrt{3}\,a$ (2-2)

$|\overrightarrow{P_1Q}| = \sqrt{(-a)^2+a^2+0^2} = \sqrt{2}\,a$

$\overrightarrow{P_1P_2} \bullet \overrightarrow{P_1Q} = -a\cdot(-a) + a\cdot a + a\cdot 0 = 2a^2.$

So $\cos\theta = \dfrac{2\cancel{a^2}}{\sqrt{3}\cancel{a}\cdot\sqrt{2}\cancel{a}} = \dfrac{2}{\sqrt{3}\cdot\sqrt{2}} = \dfrac{\sqrt{2}}{\sqrt{3}}.$

$\theta = \arccos\sqrt{\frac{2}{3}} \approx 35^\circ.$

(See also Ex. 3 in book).

② Dot product & length

$$\boxed{\vec{u}\bullet\vec{u} = u_1\cdot u_1 + u_2\cdot u_2 + u_3\cdot u_3 = u_1^2+u_2^2+u_3^2 = |\vec{u}|^2.}$$

For unit coord. vectors: $\underbrace{|\vec{i}|^2}_{\vec{i}\bullet\vec{i}} = |\vec{j}|^2 = |\vec{k}|^2 = 1$ etc.

③ Dot product & orthogonal vectors

(A) $\rightarrow$ $\vec{u}\bullet\vec{v} = |\vec{u}|\cdot|\vec{v}|\cdot\cos\theta \Rightarrow$

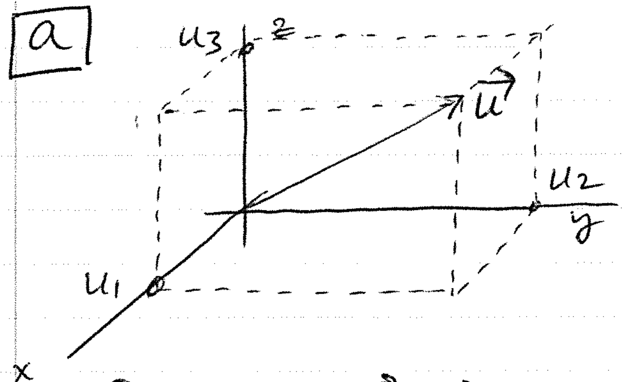
$$\boxed{(\vec{u}\perp\vec{v}) \iff \vec{u}\bullet\vec{v} = 0}$$

This is a practical TEST if $\vec{u}\perp\vec{v}$.

In particular: $\boxed{\vec{i}\bullet\vec{j} = \vec{j}\bullet\vec{k} = \vec{k}\bullet\vec{i} = 0}$.

④ Dot product & projections

a



Let's compute

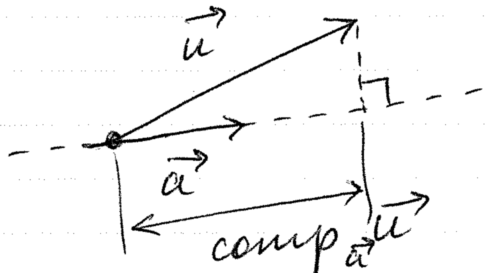
$\vec{u} \bullet \vec{i} =$

$\langle u_1, u_2, u_3 \rangle \bullet \langle 1, 0, 0 \rangle$

$= u_1 \cdot 1 + u_2 \cdot 0 + u_3 \cdot 0 = u_1.$

So: $\left\{ \begin{array}{l} \vec{u} \bullet \vec{i} = u_1 \\ \vec{u} \bullet \vec{j} = u_2 \\ \vec{u} \bullet \vec{k} = u_3 \end{array} \right\}$ ← scalar projections of $\vec{u}$ on coord. axes.

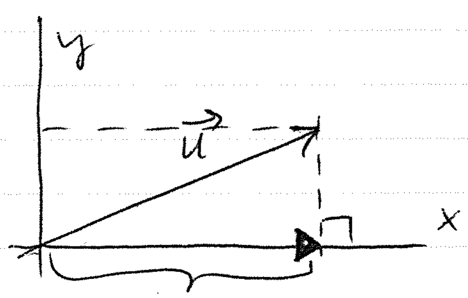
also called "components"



More generally, component of $\vec{u}$ on <u>any</u> vector $\vec{a}$:

$$\boxed{\mathrm{comp}_{\vec{a}} \vec{u} = \vec{u} \bullet \underbrace{\vec{a}^{\,*}}_{\text{unit vector along } \vec{a}} = \frac{\vec{u} \bullet \vec{a}}{|\vec{a}|}}$$

b



$u_1 \cdot \vec{i} = \mathrm{proj}_{\vec{i}} \vec{u}$ — <u>vector</u> projection of $\vec{u}$ on $\vec{i}$ (or on x-axis).

Similarly, $u_2 \cdot \vec{j} = \mathrm{proj}_{\vec{j}} \vec{u}$.

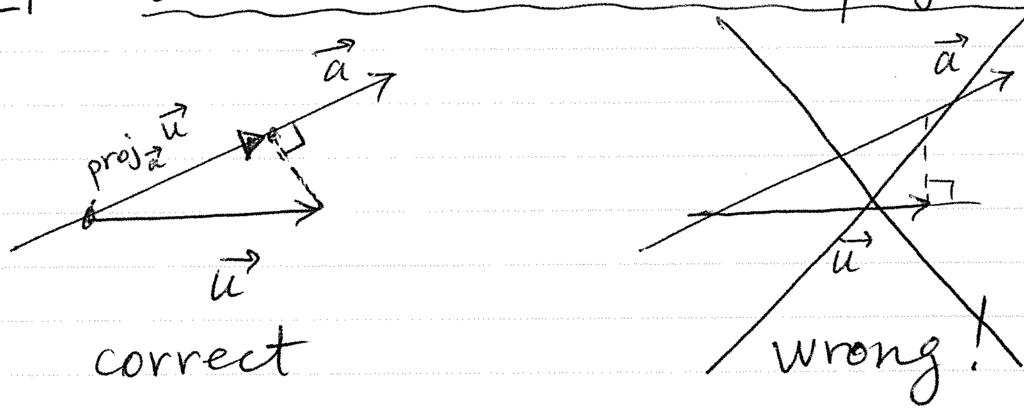
$$\text{proj}_{\vec{i}}\vec{u} = u_1 \cdot \vec{i} = (\vec{u} \circ \vec{i}) \cdot \vec{i}.$$

$$\text{proj}_{\vec{a}}\vec{u} = (\vec{u} \circ \vec{a}^*) \cdot \vec{a}^* = \frac{\vec{u} \circ \vec{a}}{|\vec{a}|} \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{(\vec{u} \circ \vec{a}) \cdot \vec{a}}{|\vec{a}|^2}.$$

arbitrary vector

Conclusion: the main use of dot product is to compute projections!

C Common mistake about projections:



Projection is always <u>SHORTER</u> than the original vector!

~~HW 3: TF @ end of Ch. 12 (p. 842): 1, 3, 19, 8~~

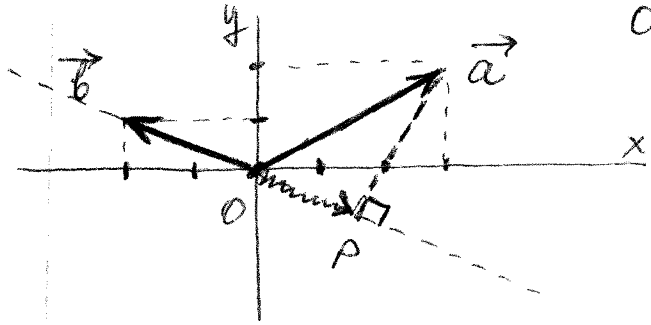
~~Sec. 12.3: 1, 14 – def.;~~
~~7, 9 – basic properties;~~
~~20, 15 – angle between vectors~~
~~23 – $\parallel$ or $\perp$~~
~~40, 41, 48 – projections.~~

~~Review Ex. p. 842 #9 – angle/diags in cube~~

~~EC #1 Sec. 12.3 #58.~~

Ex. 2 Find the scalar and vector projections of $\vec{a} = \langle 3, 2 \rangle$ on (the direction of) $\vec{b} = \langle -2, 1 \rangle$.

Sol'n:



0) In the sketch:

- $|\vec{OP}| = \text{comp}_{\vec{b}}\, \vec{a}$
- $\vec{OP} = \text{proj}_{\vec{b}}\, \vec{a}$

1) $|\vec{OP}| = \text{comp}_{\vec{b}}\, \vec{a} = \vec{a} \bullet \vec{b}^* = \dfrac{\vec{a} \bullet \vec{b}}{|\vec{b}|}$

$\vec{a} \bullet \vec{b} = \langle 3,2 \rangle \bullet \langle -2,1 \rangle = 3\cdot(-2) + 2\cdot 1 = -4$

$|\vec{b}| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$.

So, $\text{comp}_{\vec{b}}\, \vec{a} = -4/\sqrt{5} = |\vec{OP}|$.

2) $\vec{OP} = (\text{comp}_{\vec{b}}\, \vec{a}) \cdot \vec{b}^* = \dfrac{-4}{\sqrt{5}} \cdot \dfrac{\langle -2, 1 \rangle}{\sqrt{5}}$

$= \dfrac{\langle 8, -4 \rangle}{5}$.