# Review of Cross-Product (Sec. 12.4) 

## ① Definition

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

MUST know Eq. $\boxed{6}$ on p. 856 (9th Ed.)

Note: cross-product is a vector!!!

## ② Main properties of cross-product

1) $\vec{v} \times \vec{u} = - \, \vec{u} \times \vec{v}$

2) Corollary of 1):

$\vec{u} \times \vec{u} = \vec{0}$. More generally:

$$\vec{u} \times (\underbrace{k}_{\text{any scalar}}\vec{u}) = k \cdot \cancelto{\vec{0}}{(\vec{u} \times \vec{u})} = \vec{0}.$$

But $k\vec{u} \parallel \vec{u}$ (Sec. 12.2). Thus:

$$\boxed{(\vec{u} \times \vec{v} = \vec{0}) \iff (\vec{u} \parallel \vec{v})}$$

think of this as $k\vec{u}$.

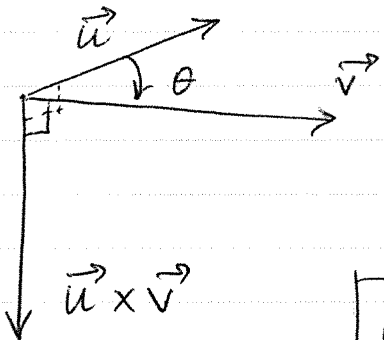
- This formula can be used to check if $\vec{u} \parallel \vec{v}$ when we don't "see" components of $\vec{u}$ & $\vec{v}$ as numbers.
- (when we see them as numbers, we just do an inspection if $\vec{v} = k \cdot \vec{u}$ for some $k$).

→ Other properties: **must** see p. 859 (e-book). (9th Ed.)

## ③ Geometric meaning & purpose of cross-product



direction: "right-hand rule"

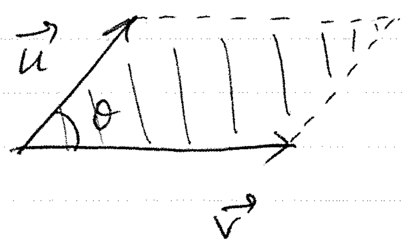



$$\boxed{\begin{array}{c}(\vec{u} \times \vec{v}) \perp \vec{u} \\ (\vec{u} \times \vec{v}) \perp \vec{v}\end{array}}$$

Thus,

$$\boxed{(\vec{u} \times \vec{v}) \perp \left(\begin{array}{l}\text{plane made} \\ \text{by } \vec{u} \;\&\; \vec{v}\end{array}\right)}$$

The main use of cross-product:
it is a vector that is $\perp$ to two given vectors.




$$\underbrace{|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \cdot \sin\theta}$$

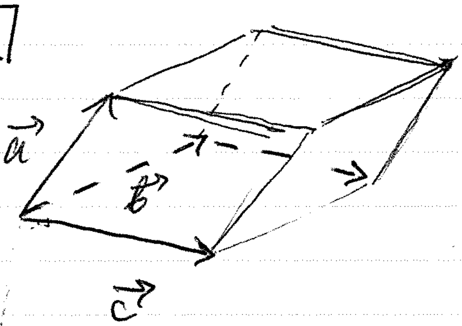
(magnitude of $\vec{u} \times \vec{v}$)

=

(area of parallelogram made by $\vec{u}$ & $\vec{v}$.)

## ④ Triple product

$\boxed{a}$



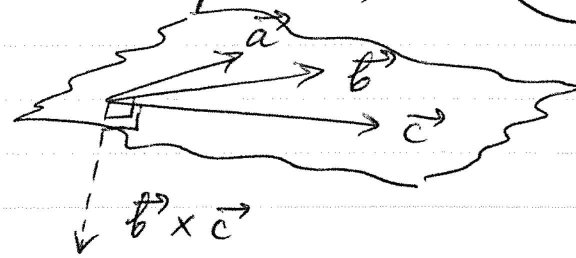

$$V = |\underbrace{\vec{a} \bullet (\vec{b} \times \vec{c})}_{\text{in any permutation}}|$$

$V$: volume of slanted box made by $\vec{a}, \vec{b}, \vec{c}$

Formula: $\vec{a} \bullet (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

3-3

**[b]** If $\vec{a}, \vec{b}, \vec{c}$ lie in same plane, then $V = 0$:




(We can also see that in this case $\vec{a} \perp (\vec{b} \times \vec{c})$,

$\Rightarrow \vec{a} \bullet (\vec{b} \times \vec{c})$ must $= 0$ ← see Sec. 12.3.)

Thus:

$$\boxed{\begin{pmatrix} \vec{a}, \vec{b}, \vec{c} \text{ lie in} \\ \text{same plane} \end{pmatrix} \Longleftrightarrow \left( \vec{a} \bullet (\vec{b} \times \vec{c}) = 0 \right)}$$

See Ex. 5 in book for numbers.

**[C]** In Lecture 1, topic 2c we showed:

$$\begin{pmatrix} \vec{a}, \vec{b}, \vec{c} \text{ lie in same plane} \\ (\text{and } \vec{a} \not\parallel \vec{b}) \end{pmatrix} \Rightarrow \begin{pmatrix} \vec{c} = s\vec{a} + t \cdot \vec{b} \\ \text{for some} \\ \text{scalars } s, t \end{pmatrix}$$

Now we can show $\Leftarrow$.

Proof: Let $\vec{c} = s\vec{a} + t\vec{b}$.
To test if $\vec{c}$ is in same plane with $\vec{a}$ & $\vec{b}$, compute triple prod:

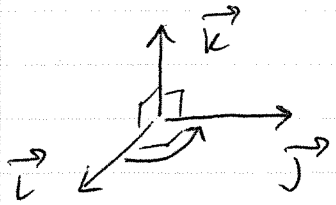
$$\vec{c} \bullet (\vec{a} \times \vec{b}) = (s\vec{a} + t\vec{b}) \bullet (\vec{a} \times \vec{b})$$

$$= s \cdot \vec{a} \bullet \underbrace{(\vec{a} \times \vec{b})}_{\perp \vec{a}} + t \cdot \vec{b} \bullet \underbrace{(\vec{a} \times \vec{b})}_{\perp \vec{b}}$$

$$= s \cdot 0 + t \cdot 0 = 0. \checkmark$$

So, $\vec{c}$ is in same plane as $\vec{a}, \vec{b}$. q.e.d.

## 5 Cross-product of $\vec{i}, \vec{j}, \vec{k}$.




$$|\vec{i} \times \vec{j}| = \underbrace{|\vec{i}|}_{1} \cdot \underbrace{|\vec{j}|}_{1} \underbrace{\sin\theta}_{\sin 90^\circ = 1.} = 1.$$

Direction: $(\vec{i} \times \vec{j}) \perp$ xy-plane, points up.

Thus,

$$\boxed{\begin{aligned} \vec{i} \times \vec{j} &= \vec{k} \\ \vec{j} \times \vec{k} &= \vec{i} \\ \vec{k} \times \vec{i} &= \vec{j} \end{aligned}}$$

(so $\vec{i} \times \vec{k} = -\vec{j}$)

Also, $\vec{i} \times \vec{i} = \vec{0} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k}$.
(see topic 2 above).

HW #3 TF (p. 842) 4, 6, 7, 9, 13, 14, 20, 21.

Sec. 12.4:
- 13, 3 — meaning, def.
- 11 — basic properties
- 16 — r.h.-rule
- 29, 19 — vector $\perp$ to two given ones
- 37, 38 — coplanar