

Lecture 8 An application of nonlinear DEs: the Logistic model

① Population with limited resources

For a population with unlimited resources (Lecture 4), we had

$$\frac{dP}{dt} = r_b P - r_d P$$

↑
birth rate was independent of P .

However, when members of the populations begin competing for food or living space, one can write:

$$r_b = r_0 (1 - kP)$$

↑ some const > 0 .

Then

$$\begin{aligned} \frac{dP}{dt} &= \underbrace{r_0(1-kP)}_{r_b} P - r_d P \\ &= (r_0 - r_d) P - r_0 k P^2 \quad \equiv aP - bP^2. \\ &= a\left(1 - \frac{b}{a}P\right)P. \end{aligned}$$

↑ ↑
some positive constants.

In notations of the textbook:

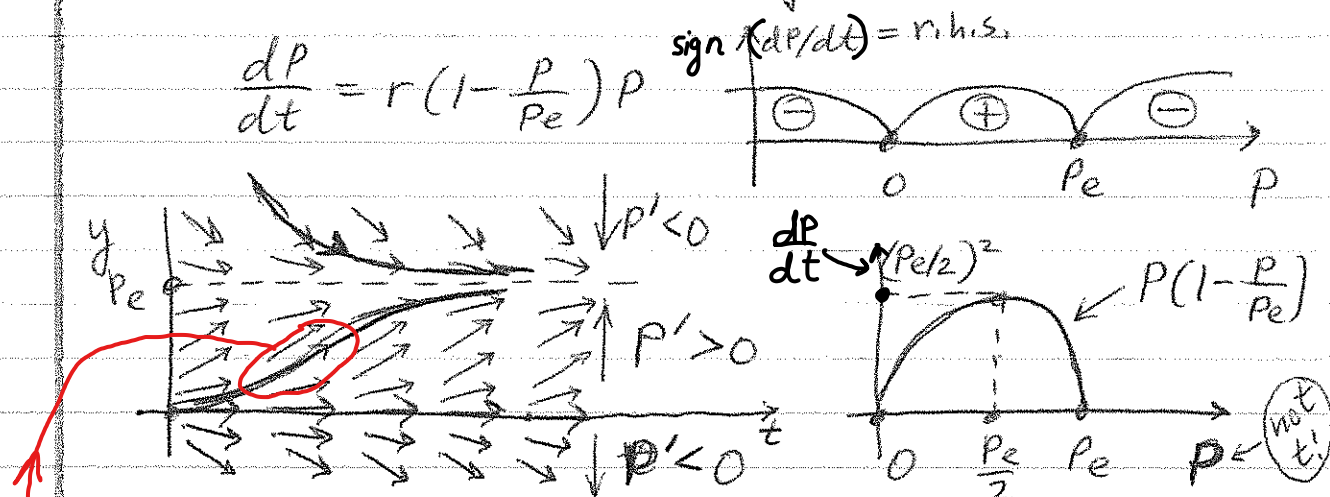
$$\frac{dP}{dt} = r\left(1 - \frac{P}{P_e}\right)P \quad (1)$$

Note: Equilibrium solutions are for $dP/dt = 0$

$$\Rightarrow P = 0 \text{ or } 1 - P/P_e = 0 \Rightarrow P = P_e$$

↑ "equilibrium".

② Qualitative behavior of solution



So, as $t \rightarrow \infty$, $P(t) \rightarrow P_e$ for any $P(0) > 0$.

There is a mistake in this trajectory. Where?

③ Exact analytical solution.

You solved the DE $\frac{dP}{dt} = P(a - P)$ at least twice in HW5/Sec. 2.6 and then also, by another method, in HW7/Sec. 2.5.

Also, the solution is presented in detail on pp. 71-72. **MUST READ IT.**

Result:

$$P(t) = \frac{P(0) \cdot P_e}{P(0) - (P(0) - P_e)e^{-rt}} \quad (2)$$

Let us verify the behavior for $t \rightarrow \infty$:

$$e^{-rt} \rightarrow 0 \Rightarrow$$

$$P(t) \rightarrow \frac{P(0) P_e}{P(0) - 0} = P_e.$$

✓ as predicted earlier

④ Constant migration

$$P' = r(1-P)P + M$$

↑
constant migration.

$M > 0$ — people move in
 $M < 0$ — people move out

Note: A more general form would be

$$P' = r\left(1 - \frac{P}{P_*}\right)P + M$$

↑ some const., but
not P_e any more!

but we will set $P_* = 1$ for simplicity.

Ex. 1 Model with migration can be transformed into one without, and hence analyzed.

Find the qualitative behavior of the solution of:

$$P' = (1-P)P - \frac{2}{9}$$

↑ $M < 0 \Rightarrow$ emigration

Sol'n:

0) Preliminaries about a linear model.

Recall $y' = a \overset{\text{const}}{y} + b$ from Lecture 2.

$$y' = a\left(y + \frac{b}{a}\right) \Rightarrow$$

-c

(3)

since
 $c' \neq 0$

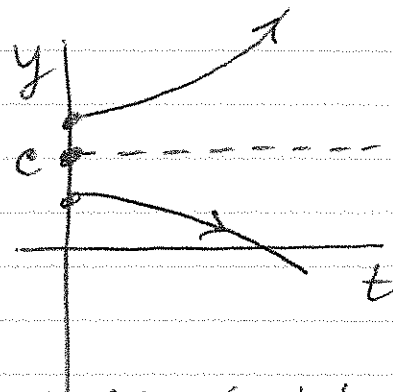
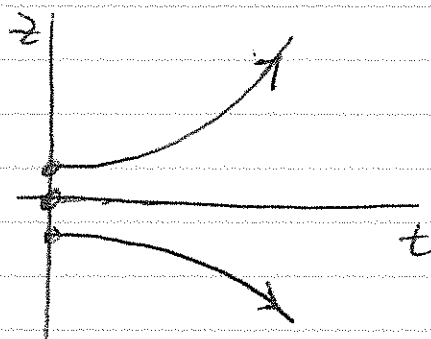
$$y' = a(y-c)$$

$$\underbrace{(y-c)}_z' = a \underbrace{(y-c)}_z$$

$$z' = az.$$

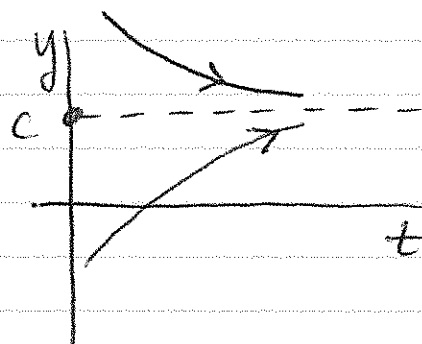
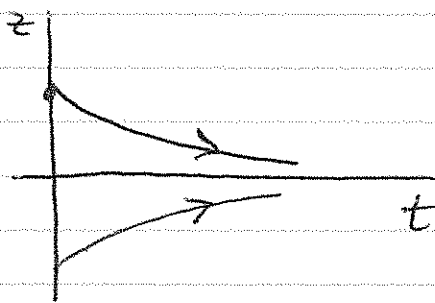
$$a > 0$$

$$z = Ce^{at}$$



(The only) Equilibrium $y=c$ is unstable (solutions moves away from it).

$$a < 0$$



(The only) Equilibrium $y=c$ is stable (all solutions move towards it).

1) Come back to our nonlinear problem.

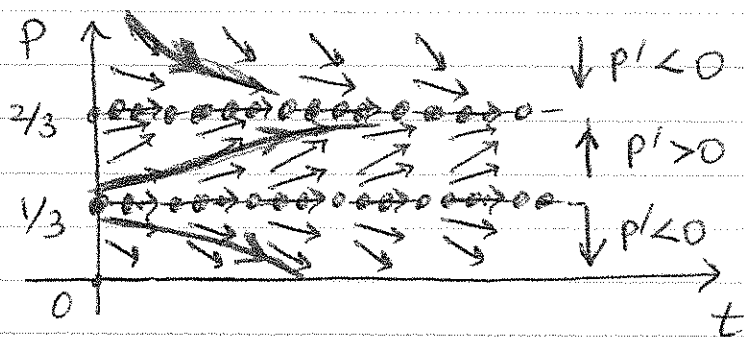
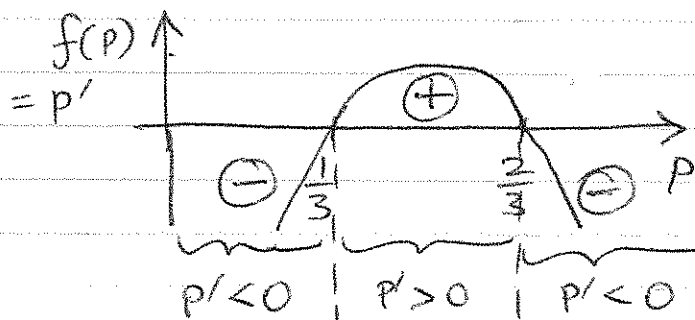
$$P' = (1-P)P - \frac{2}{9} = P - P^2 - \frac{2}{9} \equiv f(P)$$

$$= -\left(P - \frac{1}{3}\right)\left(P - \frac{2}{3}\right)$$

roots of
 $f(P)$

quadratic
function

Method 1 of analysis: look at (sign of $f(p)$)
 $=$ (sign of P').



We find:

- there are two equilibria, $(p_e)_1 = \frac{1}{3}$, $(p_e)_2 = \frac{2}{3}$.
 (Multiple equilibria can occur only in nonlinear models, but not in a linear one.)

- $(p_e)_1 = \frac{1}{3}$ is the unstable equilibrium (solutions move away from it)
- $(p_e)_2 = \frac{2}{3}$ is the stable equilibrium (solutions that are sufficiently close to it ($\frac{1}{3} < p(0) < \infty$) move towards it).

Existence of stable and unstable equilibria is common for nonlinear models.

8-6

Method 2 of analysis:

Consider behavior close to $(Pe)_1$ & $(Pe)_2$

$$P' = -\left(P - \frac{1}{3}\right)\left(P - \frac{2}{3}\right).$$

a) Let $P \approx (Pe)_2 = \frac{2}{3}$

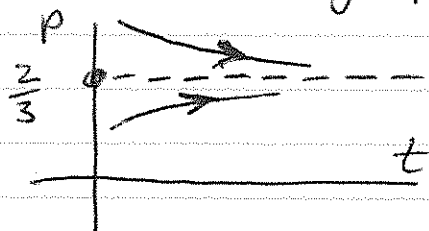
$$P' = -\left(\underbrace{\approx \frac{2}{3}}_P - \frac{1}{3}\right) \cdot \left(P - \frac{2}{3}\right)$$

$$\ominus \cdot \oplus \left(= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}\right), \text{ so } \left(-1 \cdot \frac{1}{3}\right)$$

So, near $(Pe)_2 = \frac{2}{3}$, $P' \approx -\frac{1}{3} \cdot \left(P - \frac{2}{3}\right)$

This is Eq. (3) with $a \approx -\frac{1}{3} < 0$,

and then by p. 8-4, $Pe = \frac{2}{3}$ is stable.



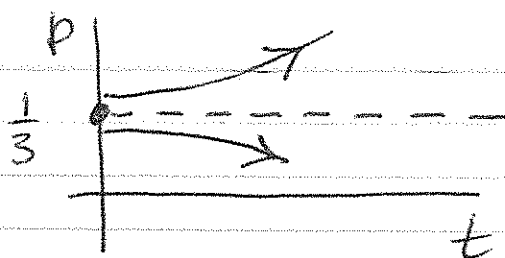
} all solutions that are sufficiently close to $Pe = \frac{2}{3}$ tend towards it for $t \rightarrow \infty$.

b) Let $P \approx (Pe)_1 = \frac{1}{3}$

$$P' = -\left(\underbrace{P - \frac{2}{3}}_{\approx -\frac{1}{3}}\right)\left(P - \frac{1}{3}\right)$$

$$\ominus \cdot \ominus \left(= \frac{1}{3} - \frac{2}{3} = -\frac{1}{3}\right), \text{ so } -1 \cdot \left(-\frac{1}{3}\right) = \frac{1}{3}.$$

So, near $(Pe)_1 = \frac{1}{3}$, $P' \approx \left(+\frac{1}{3}\right) \cdot \left(P - \frac{1}{3}\right)$.
This is Eq. (3) with $a = +\frac{1}{3} > 0$, and then by p. 8-4, $Pe = \frac{1}{3}$ is unstable.



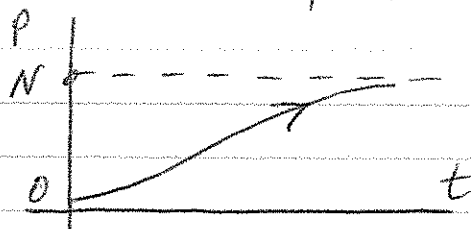
} all solutions that are sufficiently close to $p_e = 1/3$ tend away from it as $t \rightarrow \infty$.

⑤ Other applications of the logistic model

There are many.

A) Infectious disease w/o recovery (p. 74)

$$\frac{dP}{dt} = k \cdot \underset{\substack{\uparrow \\ \text{\# of infected}}}{P} \cdot \overset{\substack{\text{total} \\ \downarrow}}{(N - P)} \quad \text{\# of non-infected.}$$



Everyone will get infected eventually.

b) Quadratic drag force (pp. 79-83) ← optional.

HW, Sec. 2.8: 1, 2, 3; 4, 5, 6, 7; 18, 19.

Hint for #18: See Eq. (7) in Sec. 2.8 and use the exact solution of Eq. (1) (also in the book).

Answers: #2: $10 \ln 4.4$

#4: $p_e = 1/4, 3/4$; $p_e = 3/4$ is stable

#6: $p_e = 1/2$; $P(t) \rightarrow -\infty$