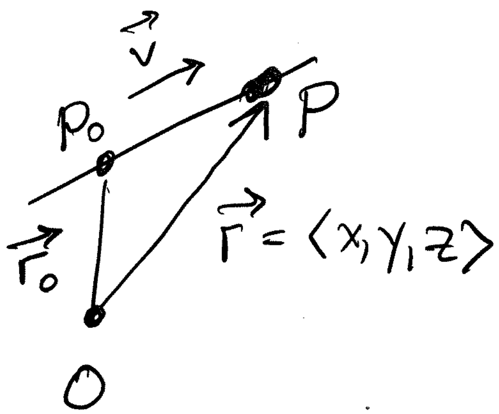
Sec. 13.1 Vector-valued functions

1. Parametric eqs. of curves in 3D

In Sec. 12.5A we learned that



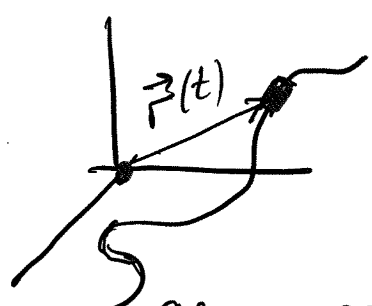
a line has the vector equation

$$\vec{r}(t) = \vec{r}_0 + \vec{v}\,t,$$

or, equivalently, parametric equations

$$x(t) = x_0 + at, \quad y(t) = y_0 + bt, \quad z(t) = z_0 + ct.$$

(You can think of these as the eqs. of motion of a car that starts to move at location $\vec{r}_0 = \vec{OP_0}$ and goes with a constant velocity $\vec{v} = \langle a, b, c \rangle$.)



Similarly, you can think of some general parametric function $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ as a road in 3D, on which a car moves.

Thus, in a parametric curve with eqs

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle,$$

the radius vector

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

gives the location of a point on that curve.

This $\vec{r}(t)$ is called a <u>vector-valued function.</u>

Another notation:

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle.$$

In 3D, a vector (-valued) function is an ordered list of 3 scalar-valued functions.

<u>Domain of a vector function $\vec{r}(t)$:</u> All values of t where all three components $x(t), y(t), z(t)$ are defined simultaneously. (See Ex. 1 in book)

(2) <u>Sketches of some vector-valued functions</u>

<u>Name:</u> $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $\Rightarrow$

$x = f(t)$, $y = g(t)$, $z = h(t)$ defines a <u>parametric curve</u>.

<u>Ex. 1</u> Sketch a parametric curve

$$x = e^t, \quad y = \cos(2t), \quad z = \sin(2t).$$

<u>Sol'n:</u>

1) First, we recognize that in the projection on the yz-plane (i.e. if we <u>ignore</u> x-coord. for now), we have a circle (see Sec. 12.6)

$$y = \cos 2t$$
$$z = \sin 2t$$

<u>Q:</u> What interval of t gives a 360° turn?

2) What happens to x as t varies?

$t=0 \Rightarrow x=e^0=1,\ y=\cos 0=1,\ z=\sin 0=0.$

$t \nearrow \Rightarrow x \nearrow$ exponentially to ∞.

$t \searrow \Rightarrow x=e^{-t}$; goes to 0.

The curve will wrap around the cylinder $y^2+z^2=1$ and will asymptotically approach the plane $x=0$, making infinitely many turns.

In the other direction along t, the curves moves away to $x=\infty$ while staying on the cylinder $y^2+z^2=1$.

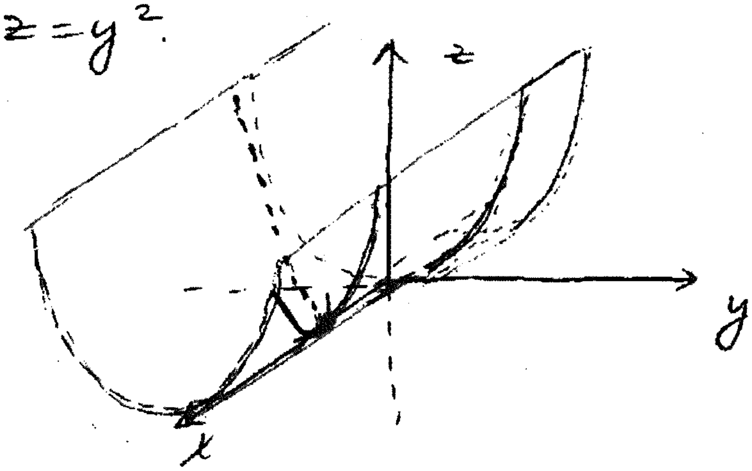
Some Helix's applications:
- DNA
- Electron with $\vec{v} \parallel \vec{B}$.

(Compare with Ex. 4 in book.) MUST READ to learn about a HELIX

Ex. 2 Sketch the curve $\vec{r}(t)=\langle e^t, t, t^2\rangle$.

Sol'n: 1) In yz-plane we now have a parabola: $y=t,\ z=t^2 \Rightarrow z=y^2$:

So the curve will lie on the cylinder $z=y^2$.

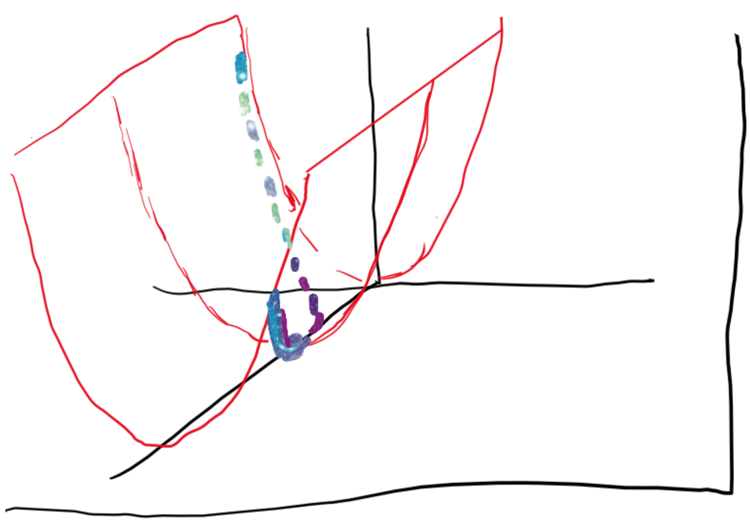


2) Plot the curve on this cylinder by considering some values of t.

$\underline{t=0} \Rightarrow x=e^0=1,\ y=0,\ z=0$

$\underline{t\uparrow} \Rightarrow x\uparrow$ exponentially towards $+\infty$

$t\searrow \Rightarrow x=e^{-t}\to 0$ (but is >0).



Note: This curve also belongs to the cylinder

$x=e^t,\ y=t$

or $\quad x=e^y$.

Then, this curve is the intersection of two cylinders, $z=y^2$ & $x=e^y$.

Thus, any 3D curve is the intersection of some two cylinders. $\boxed{\text{So, any 3D curve is defined by 2 equations relating } x, y, z\,!}$

3 A curve lying on a circle

7-5

Ex. 3 Show that the parametric curve
$x = t^3\cos t$, $y = t^3 \sin t$, $z = t^3$
lies on a cone. Use this fact to sketch the curve.

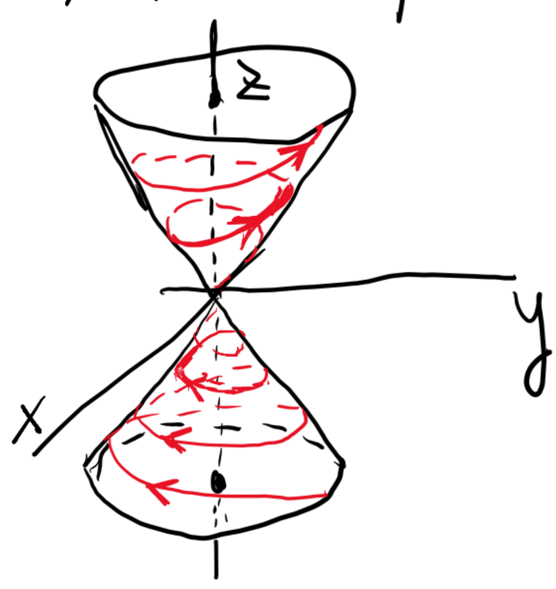
Sol'n: 1) In HW 6 (## 43, 15) you learned that the equation of a cone relates x^2, y^2, & z^2.

$x^2 = (t^3\cos t)^2 = t^6\cos^2 t$
$y^2 = (t^3 \sin t)^2 = t^6 \sin^2 t$
$z^2 = (t^3)^2 = t^6$.

By inspection: $t^6\cos^2 t + t^6 \sin^2 t = t^6$
(using $\cos^2 t + \sin^2 t = 1$) $\Rightarrow x^2 + y^2 = z^2$.

2) Sketch.

$z = t^3$ can be ⊕ & ⊖,
$\Rightarrow$ curve lies on both halves of the cone.



4 <u>Intersection of a parametric curve with a surface or another curve.</u>

(Compare with: Ex. 6/book + Ex. 2/Notes for Sec. 12.5B.)

<u>Ex. 4</u> At what points does the helix (see Ex. 4 in book).

$\vec{r}(t) = \cos t \cdot \vec{i} + \sin t \, \vec{j} + t \vec{k}$

intersect the sphere $x^2 + y^2 + z^2 = 4$?

<u>Sol'n:</u> All you need to do is to substitute $x(t), y(t), z(t)$ into the eq. of the surface and find for what t the eq. is true:

$$x^2 + y^2 + z^2 = 4$$

$$\underbrace{(\cos t)^2 + (\sin t)^2}_{} + (t)^2 = 4$$

$$1 + t^2 = 4 \Rightarrow t = \pm\sqrt{3}.$$

<u>Answer:</u> <u>$t = -\sqrt{3}$</u> : $P_1 = (\cos(-\sqrt{3}), \sin(-\sqrt{3}), -\sqrt{3})$

$t = \sqrt{3}$: $P_2 = (\cos\sqrt{3}, \sin\sqrt{3}, \sqrt{3})$.

<u>Discuss:</u> Intersection of two curves $\vec{r}_1(t) = \langle f_1(t), g_1(t), h_1(t) \rangle$ and $\vec{r}_2(t) = \langle f_2(t), g_2(t), h_2(t) \rangle$:

For what t and s is $\vec{r}_1(t) = \vec{r}_2(s)$?