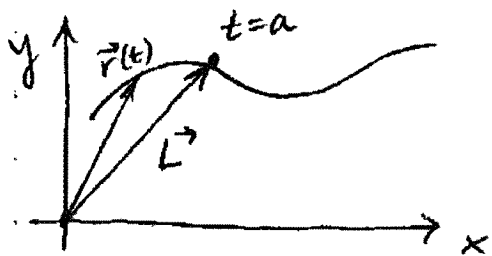


Sec. 13.2. Derivatives & integrals of vector functions.

① Limits and continuity of v.f.

Draw all pictures in 2D; conclusions apply in 3D.

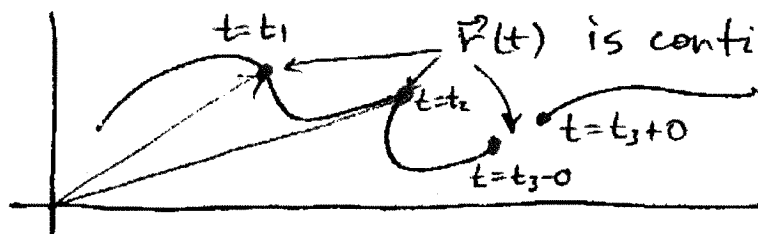


$$\lim_{t \rightarrow a} \vec{r}(t) = \vec{L}$$

if $\vec{r}(t)$ approaches $\vec{L}$ as $t \rightarrow a$.

" $\vec{r}(t) \rightarrow \vec{L}$ " means "each component of $\vec{r} \rightarrow$ the corresponding component of $\vec{L}$ ".

V.f. $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is continuous at $t = t_0$ if the param. curve $\vec{r} = \vec{r}(t)$ ($x=f(t)$, $y=g(t)$, $z=h(t)$) has no jump at $t = t_0$.



$\vec{r}(t)$ is continuous at t_1, t_2 , but discontinuous at $t = t_3$.

② Derivatives of a v.f.

Meaning:

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle \leftarrow$ location (position)

$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle \leftarrow$ velocity

$\vec{r}''(t) = \langle f''(t), g''(t), h''(t) \rangle \leftarrow$ acceleration

Differentiation rules (p. 858, 8th ed.)

1) $(\vec{u} \pm \vec{v})' = \vec{u}' \pm \vec{v}'$

2) $c = \text{const} \Rightarrow (c \cdot \vec{u})' = c \vec{u}'$

$(\vec{c} = \text{const vector}) \Leftrightarrow \vec{c}' = \vec{0}$

3) $(f(t) \cdot \vec{u})' = f' \vec{u} + f \cdot \vec{u}'$

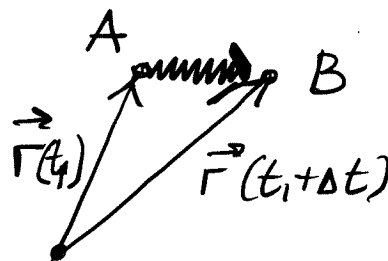
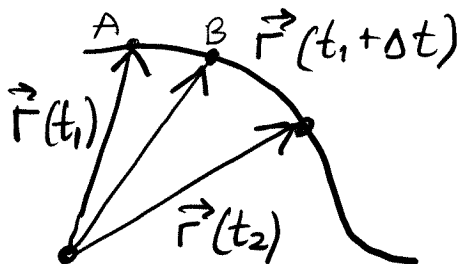
4) $(\vec{u} \bullet \vec{v})' = \vec{u}' \bullet \vec{v} + \vec{u} \bullet \vec{v}'$

$(\vec{u} \times \vec{v})' = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$

Product Rules

E.g., here is a Proof of 3) (see book for 4):

$$\begin{aligned}
 (f \cdot \vec{u})' &= (f \cdot \langle u_1, u_2 \rangle)' = \langle f u_1, f u_2 \rangle' = \\
 &= \langle (f u_1)', (f u_2)' \rangle = \\
 &= \langle \underline{f' u_1 + f u_1'}, \underline{f' u_2 + f u_2'} \rangle = \\
 &= \langle \underline{f' u_1}, \underline{f' u_2} \rangle + \langle \underline{f u_1'}, \underline{f u_2'} \rangle \\
 &= f' \vec{u} + f \vec{u}'.
 \end{aligned}$$

③ Tangent vector to a curve

$$\vec{r}(t_1) + \vec{AB} = \vec{r}(t_1 + \Delta t)$$

$$\vec{AB} = \vec{r}(t_1 + \Delta t) - \vec{r}(t_1) \text{ (see Sec. 12.2.)}$$

$$\text{As } \Delta t \rightarrow 0, \vec{AB} \rightarrow \vec{0}, \text{ but } \lim_{\Delta t \rightarrow 0} \frac{\vec{AB}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t} = \vec{r}'(t).$$

Thus, $\vec{r}'(t_1)$ is always tangent to $\vec{r}(t)$ @ $t = t_1$.

MUST READ/FOLLOW THE PROOF OF

8-3

Theorem 4 in textbook.

Discussion of meaning of Theorem 4:



"A bug is crawling on a sphere centered at the origin. Show that the bug's velocity is always $\perp$ to its position vector $\vec{r}(t)$."

Ingredients of solution:

Given: Sphere $\Rightarrow |\vec{r}(t)| = a$ $\leftarrow$ some const.

Want: velocity $\perp \vec{r}(t) \Leftrightarrow$
 $\vec{r}' \perp \vec{r}$

Steps of solution (helping you to go through Thm. 4)

- Relate the length $|\vec{r}|$ to some dot product (See Notes on Sec. 12.3).
- Use the Product Rule for the dot product (see previous page).
- Use properties of the dot product (Sec. 12.3) after you differentiate.
- Finally, from your calculation, use a criterion from Sec. 12.3 that shows that two vectors are $\perp$.

Ex. 1
Find a unit tangent vector to
 $\vec{r} = \langle \sin 3t, \cos^2 t \rangle$ at $t = \pi/3$.

Sol'n: 1) A tangent vector is $\vec{r}'(t) =$
 $\langle \sin 3t, \cos^2 t \rangle' =$
 $= \langle (\sin 3t)', (\cos^2 t)' \rangle$

Absolutely
MUST DO!

Chain Rule reminder

[Review Sec. 3.4 on your own!]

$$\begin{aligned} \frac{d}{dt}(\sin 3t) &= \frac{d}{dt} \sin u(t) = \frac{d \sin u}{du} \cdot \frac{du}{dt} \quad \left| \begin{array}{l} u(t) = 3t \end{array} \right. \\ &= \cos u \cdot \frac{d(3t)}{dt} = \cos(3t) \cdot 3 \\ &= 3 \cos 3t. \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}(\cos^2 t) &= \frac{d u^2}{dt} = \frac{d u^2}{du} \cdot \frac{du}{dt} \quad \cos t = u \\ &= 2u \cdot \frac{d(\cos t)}{dt} = 2 \cos t \cdot (-\sin t) = -2 \cos t \cdot \sin t. \end{aligned}$$

Thus, $\vec{r}'(t) = \langle 3 \cos 3t, -2 \cos t \cdot \sin t \rangle$

2) Unit tan. vector @ $t = t_0 = \pi/3$:

$$\begin{aligned} \vec{r}'(t_0) &= \langle 3 \cos 3t_0, -2 \cos t_0 \cdot \sin t_0 \rangle \\ &= \langle 3 \cos \pi, -2 \cos^{\frac{1}{2}} \frac{\pi}{3} \cdot \sin^{\frac{\sqrt{3}}{2}} \frac{\pi}{3} \rangle = \langle -3, -\sqrt{3}/2 \rangle. \end{aligned}$$

Recall from Ex. 4 in Notes for Sec. 12.2 that
any vector $\vec{u}_* = \frac{\vec{u}}{|\vec{u}|}$ has unit length.

Thus, $\frac{\vec{r}'(t_0)}{|\vec{r}'(t_0)|}$ is a unit tangent vector

8-5

to curve $\vec{r} = \vec{r}(t)$ at $t = t_0$.

$$|\vec{r}'(t_0)| = \sqrt{(-3)^2 + (-\sqrt{3}/2)^2} = \sqrt{39/4}.$$

Answer: $-\frac{\langle 3, \sqrt{3}/2 \rangle}{\sqrt{39/4}}$

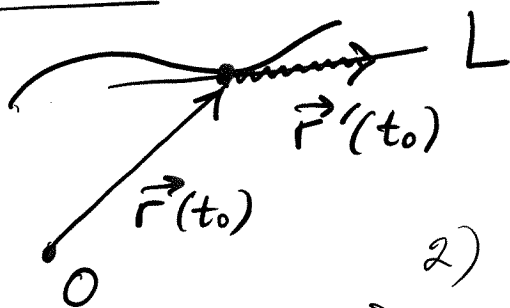
Generalization:

$$\boxed{\vec{T}(t) \equiv \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \text{ is the unit tangent vector to curve } \vec{r} = \vec{r}(t).}$$

Ex. 2 (see also Ex. 3 in book)

Find the parametric eqs. for the tangent line to the curve of Ex. 1 at $t_0 = \pi/3$.

Sol'n:



1) We need 2 ingredients of a line:

- A point on it;
- A vector along it.

2) A point on it is

$$\vec{r}(t_0)|_{t_0=\pi/3} = \langle \sin(3 \cdot \frac{\pi}{3}), \cos^2 \frac{\pi}{3} \rangle = \langle 0, 1/4 \rangle.$$

(Ex. 1)

$$3) \vec{r}'(t_0)|_{t_0=\pi/3} = \langle -3, -\sqrt{3}/2 \rangle$$

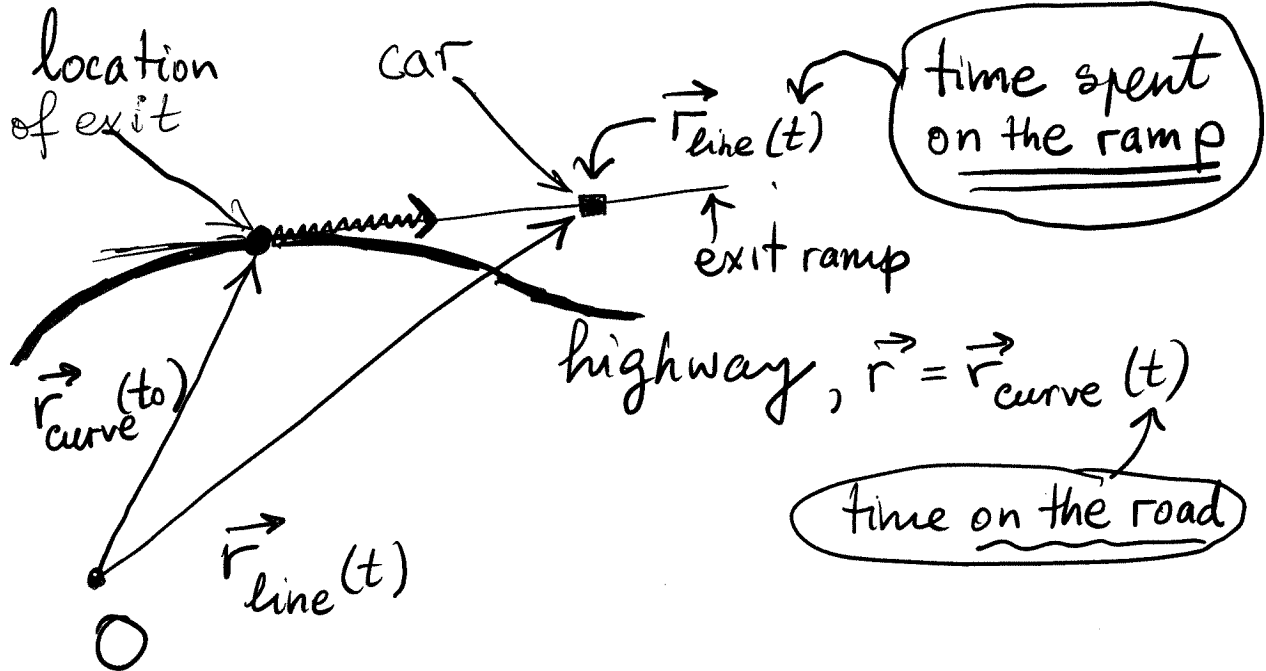
Note: We don't need to make it a unit vector.

$$4) \text{ Vector eq. of the tan. line: } \vec{r}_{\text{line}}(t) = \vec{r}_{\text{curve}}(t_0) + \vec{r}'_{\text{curve}}(t_0) \cdot t$$

Parametric eqs: $\langle x, y \rangle_{\text{line}} = \langle 0, 1/4 \rangle + \langle -3, -\frac{\sqrt{3}}{2} \rangle \cdot t$

Highway - exit analogy for the tangent line to a curve :

car exits at $t=t_0$ (time measured on the road)



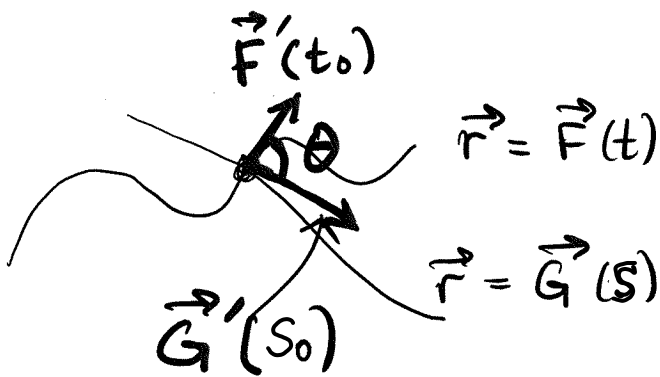
time on exit ramp

$$\underbrace{\vec{r}_{\text{line}}(t)}_{\text{location on exit ramp}} = \underbrace{\vec{r}_{\text{curve}}(t_0)}_{\text{location of exit}} + \underbrace{\vec{r}'_{\text{curve}}(t_0)}_{\text{velocity at exit}} \cdot t$$

time when the car arrives at the exit on highway

Note about the angle between two intersecting curves:

8-7



If curves $\vec{r} = \vec{F}(t)$
and $\vec{r} = \vec{G}(s)$
intersect at some
point where
 $\vec{F}(t_0) = \vec{G}(s_0)$,

then the angle between them
is defined as the angle between their tangent vectors
at the intersection point. See Sec. 12.3.

④ Integrals of vector functions

Idea: Simply integrate each component of the vector function (see Ex. 5 in the book).

Ex. 3 Find $\vec{r}(t)$ if $\vec{r}'(t) = \vec{i} \cdot \cos t + \vec{j} \cdot \sin t$
and $\vec{r}(\pi/2) = 3\vec{i} - \vec{j}$.

Interpretation: Find the location of a car ($\vec{r}(t)$) at all times if you know its velocity ($\vec{r}'(t)$) and location at some instance of time ($t_0 = \pi/2$).

Sol'n:

$$1) \vec{r}(t) = \int \vec{r}'(t) dt = \int \langle \cos t, \sin t \rangle dt = \\ = \langle \int \cos t dt, \int \sin t dt \rangle = \langle \sin t + C_1, -\cos t + C_2 \rangle.$$

Note that the constants for each component are different.

2) Find C_1, C_2 .

$$\text{At } t = \pi/2, \vec{r}(\pi/2) = \langle \sin \frac{\pi}{2} + C_1, -\cos \frac{\pi}{2} + C_2 \rangle \\ = \langle 1 + C_1, C_2 \rangle.$$

On the other hand, we are given that

$$\vec{r}(\pi/2) = \langle 3, -1 \rangle. \quad \text{Thus:}$$

$$\langle 1 + C_1, C_2 \rangle = \langle 3, -1 \rangle \Rightarrow$$

$$\begin{cases} 1 + C_1 = 3 \Rightarrow C_1 = 2 \\ C_2 = -1 \end{cases}$$

Answer: $\vec{r}(t) = \langle (\sin t) + 2, (-\cos t) - 1 \rangle.$