

Sec. 14.5: Chain Rule

15-1

① Explicit differentiation

Recall from Calc. I:

If $z = f(x)$ and $x = x(t)$, then

$$\frac{dz}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} \quad \leftarrow \begin{array}{l} \text{Calc. I} \\ \text{Chain Rule} \end{array}$$

Our goal in this topic will be to **extend** this 1-variable rule (i.e., each of f and x depend only on 1 variable) **to the multi-variable case.**

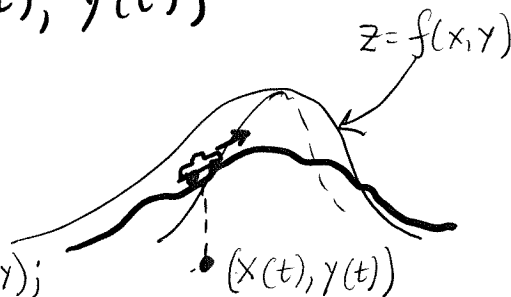
Case 1(a): $z = f(x(t), y(t))$

Interpretation:

Car on the road;

Road is on the hill $z = f(x, y)$;

In (x, y) -plane, the car's position is given by $(x(t), y(t))$.



One may then ask: How does the elevation (i.e. z -coordinate) of the car change with t ?

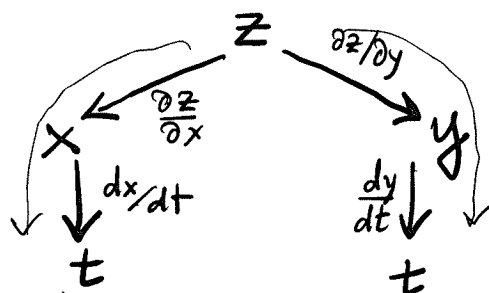
Formula:

$$\frac{dz}{dt} \equiv \frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

"same as"

See Ex. 1/book
for numbers.

Tree diagram:



Comments:

1) Always write the Tree Diagram before writing out the formula.

In the Tree Diagram, trace all paths that lead from z to t (2 such paths in the case above).

2) Note: $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ are partial derivatives, but $(\frac{dx}{dt}, \frac{dy}{dt})$ are ordinary derivatives. Why?

Because $f(x, y)$ depends on more than one variable, $\Rightarrow$ need partials.

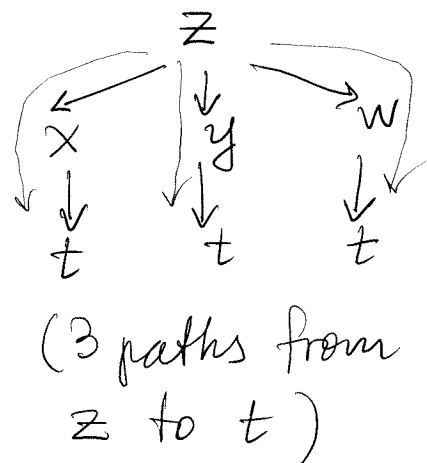
On the other hand, each of $x(t), y(t)$ depends on one variable, $\Rightarrow$ can only have ordinary derivative.

3) If f depends on more than 2 variables, the formula is similar:

$$\frac{d f(x(t), y(t), w(t))}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial w} \cdot \frac{dw}{dt}$$

etc.

Tree Diagram:



Case 1(b): In the last formula
on the previous page,

let's set $w(t) = t$. Then we have:

$$\frac{d f(x(t), y(t), t)}{d t} = \frac{\partial f}{\partial x} \cdot \frac{d x}{d t} + \frac{\partial f}{\partial y} \cdot \frac{d y}{d t} + \frac{\partial f}{\partial w} \cdot \frac{d w}{d t}$$

but: $w = t$ $\frac{d t}{d t} = 1$

Thus:

$$\boxed{\frac{d f(x(t), y(t), t)}{d t} = \frac{\partial f}{\partial x} \cdot \frac{d x}{d t} + \frac{\partial f}{\partial y} \cdot \frac{d y}{d t} + \frac{\partial f}{\partial t}}$$

Ex. 1 $f(x, y, t) = t x^2 + \sin(xy)$, $x = 3\sqrt{t}$, $y = e^{5t}$.

Find df/dt .

Sol'n: Use the formula of Case 1(b):

$$\begin{aligned} \frac{d f}{d t} = & \underbrace{\left(t \cdot 2x + y \cdot \cos(xy) \right)}_{\partial f / \partial x} \cdot \underbrace{\frac{3}{2\sqrt{t}}}_{d x / d t} + \\ & + \underbrace{x \cdot \cos(xy)}_{\partial f / \partial y} \cdot \underbrace{5e^{5t}}_{d y / d t} + \underbrace{x^2}_{\partial f / \partial t} \end{aligned}$$

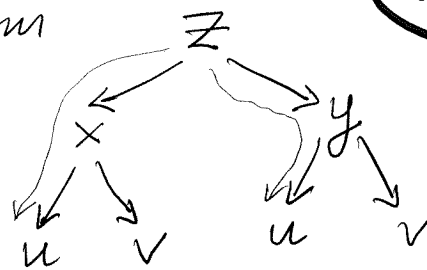
(Can simplify, but this won't be required on quiz/test.)

Case 2 $z = f(x, y)$, $x = x(u, v)$
 $y = y(u, v)$

To obtain the formula for, say, $\frac{\partial z}{\partial u}$, draw
the Tree Diagram.

There are 2 paths leading from z to u . So:

$$\boxed{\frac{\partial f(x(u,v), y(u,v))}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}}$$



and similarly for $\partial f / \partial v$.

Note: All derivatives here are partial (not ordinary) because all variables: f, x, y depend on more than 1 variable (see the Tree Diagram).

Ex. 2 (similar to Ex. 3 in book):

Let $z = x^5 e^{y^2}$, $x = r \cos \theta$, $y = r \sin \theta$.

Find $\partial z / \partial \theta$.

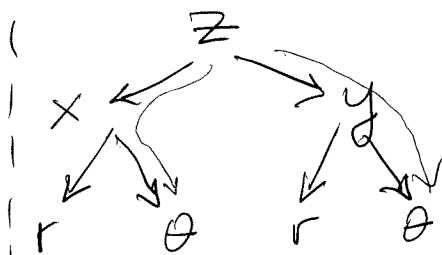
Sol'n:

$$\frac{\partial z}{\partial \theta} \overset{\text{from Tree}}{=} \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$= (5x^4 \cdot e^{y^2}) \cdot (-r \sin \theta) +$$

$$(x^5 \cdot 2y \cdot e^{y^2}) \cdot (r \cos \theta).$$

Tree:



One can, of course, simplify this expression, but this will not be required on test/quiz.

Case 2 for functions of 3 or more variables is handled similarly; See Ex. 4, 5/book.

② Implicit differentiation.

Review of Calc. I situation (recall that you were required to read Ex. 5 in Sec. 14.3 to review this material):

Find the slope of circle $x^2 + y^2 = R^2$ at a given point (x, y) on the circle.

Sol'n: On the circle, we have $y = y(x)$.
Differentiate the equation of circle w.r.t. x :

$$\frac{d}{dx} (x^2 + (y(x))^2 = R^2) \Rightarrow$$

$$2x + 2y \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}.$$

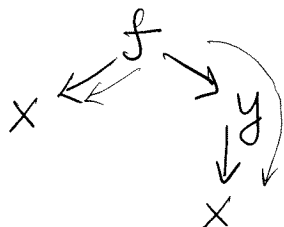
See also Ex. 8 in textbook (Sec. 14.5).

We can generalize this from a circle to any curve:

$f(x, y) = 0$ defines some curve $y = y(x)$
($x^2 + y^2 - R^2 = 0$ in the above case of the circle).

$$\frac{d}{dx} (f(x, y(x)) = 0) \Rightarrow \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$$

↑
Case 1 in topic 1



$$\Rightarrow \frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y}.$$

And now we can generalize to 3 variables:

$F(x, y, z) = 0$ defines some surface $z = z(x, y)$.

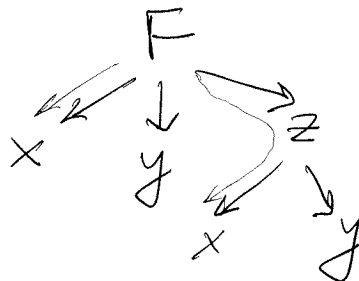
(Think of sphere $x^2 + y^2 + z^2 - R^2 = 0$)

To find the slope of this surface along, say, x -axis:

$$\frac{d}{dx} (F(x, y, z(x, y)) = 0) \Rightarrow$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0 \Rightarrow$$

$$\frac{\partial z}{\partial x} = - \frac{\partial F / \partial x}{\partial F / \partial z}$$



See posted page 13-4 and Ex. 9 in book.

③ Application to rates of change

MUST READ Ex. 2 in textbook.

④ Application to partial differential equations

Ex. 3 Show that for any function $f(u)$, the function $z = f(x + 2y)$ satisfies a partial diff. equation $2 \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$.

Interpretation (vague): This P.D. equation is a special case of the Wave Eqn. (sec. 14.3). The problem "shows" (vague!) that a disturbance of any shape can propagate along a tight string (vague!!)

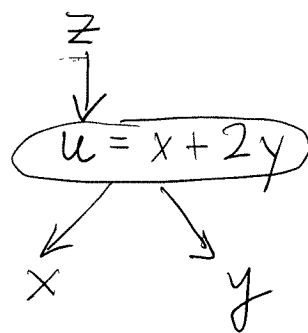
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Sol'n:

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{df}{du} \cdot \frac{\partial (x+2y)}{\partial x}$$

$$= \frac{df}{du} \cdot 1$$

$$\frac{\partial z}{\partial y} \underset{\substack{\uparrow \text{ similarly} \\ \nearrow (x+2y)}}{=} \frac{df}{du} \cdot \frac{\partial u}{\partial y} = \frac{df}{du} \cdot 2.$$



Thus,

$$2 \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 2 \cdot \frac{df}{du} \cdot 1 - \frac{df}{du} \cdot 2 = 0. \quad \checkmark$$

See also Ex. 6 / book (it is more complicated).