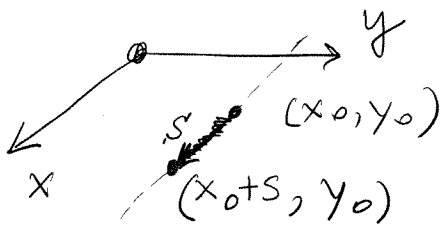
Sec. 14.6 Directional derivative and Gradient

You will learn two concepts named in the title. Both **generalize the concept of partial derivatives** in different ways. The concept of the **gradient** is **ubiquotous in science/engineering**, and we will continue to use it for the rest of the course.

(1) Directional derivative

To begin, recall the definition of x-partial derivative (= rate of change of f w.r.t. x): (Notes for Sec. 14.3; also in Lab 4.)



$$f_x(x_0, y_0) = \lim_{s \to 0} \frac{f(x_0+s, y_0) - f(x_0, y_0)}{s}$$

$$= \left. \frac{df(x_0+s, y_0)}{ds} \right|_{s=0}$$

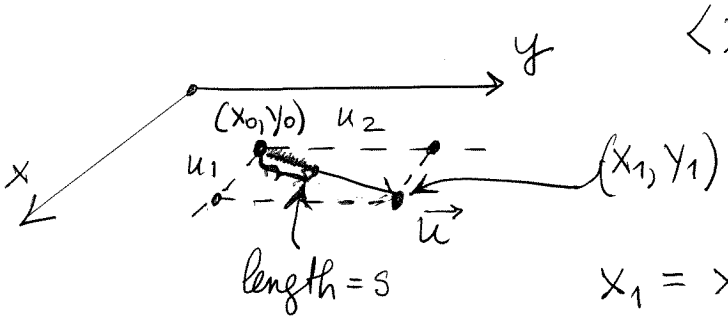
We will generalize this formula soon.

Also recall that f_x is the slope of f in the direction of the x-axis.

But why restrict oneself to considering the slope along x- or y-axes? One can consider the slope along any direction in the xy-plane!

You are NOT required to read this derivation. It's here FYI.

Let $\vec{u}=\langle u_1, u_2\rangle$ be some unit vector in the xy-plane. Let us find the rate of change of $f(x,y)$ in the direction of $\vec{u}$ similarly to how we did it for f_x on the previous page.



$\langle x_1, y_1\rangle = \langle x_0, y_0\rangle + s\cdot \vec{u}$ ($\langle u_1, u_2\rangle$)

length, direction

$x_1 = x_0 + s\cdot u_1$

$y_1 = y_0 + s\cdot u_2$

$$D_{\vec{u}} f = \lim_{s\to 0} \frac{f(x_1,y_1) - f(x_0,y_0)}{s} =$$

$D_{\vec{u}} f$: derivative of f in direction of $\vec{u}$

$$= \lim_{s\to 0} \frac{f(x_0+su_1, y_0+su_2) - f(x_0,y_0)}{s}$$

$$= \frac{d}{ds} f(x_0+su_1, y_0+su_2)\Big|_{s=0}$$

Chain Rule 14.5

$$= \left(\frac{\partial f}{\partial x}\cdot\frac{d(x_0+su_1)}{ds} + \frac{\partial f}{\partial y}\cdot\frac{d(y_0+su_2)}{ds}\right)\Big|_{s=0}$$

$$= f_x(x_0,y_0)\cdot u_1 + f_y(x_0,y_0)\cdot u_2$$

But you MUST MEMORIZE This formula.

$$D_{\vec{u}} f = f_x(x_0,y_0)\cdot u_1 + f_y(x_0,y_0)\cdot u_2$$

where: $\vec{u}=\langle u_1, u_2\rangle$, $|\vec{u}| = 1$

$D_{\vec{u}} f$ is called the **directional derivative** of f along/in direction of vector $\vec{u}$.

$D_{\vec{u}} f$ is the slope of surface $z=f(x,y)$ in direction of $\vec{u}$.

Ex. 1 Find $D_{\vec{u}} f$ for $f = x^2 y$ at point $(3,2)$ in the direction of $\vec{v} = \langle 4, 3 \rangle$.

Sol'n: 1) Since $\vec{v}$ is not a unit vector, $\Rightarrow$ find a unit vector along $\vec{v}$ (sec. 12.2):

$$\vec{v}^* = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 4, 3 \rangle}{\sqrt{4^2+3^2}} = \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle.$$

This is a unit vector

2) Use the formula:

$$D_{\vec{v}} f \equiv D_{\vec{v}^*} f = \frac{\partial}{\partial x}(x^2 y)\Big|_{(3,2)} \cdot \frac{4}{5} + \frac{\partial}{\partial y}(x^2 y)\Big|_{(3,2)} \cdot \frac{3}{5}$$

it does not matter which one we list, since they have the same direction

$$= 2xy\Big|_{(3,2)} \cdot \frac{4}{5} + x^2\Big|_{(3,2)} \cdot \frac{3}{5}$$

$$= (2 \cdot 3 \cdot 2) \cdot \frac{4}{5} + 3^2 \cdot \frac{3}{5} = 15$$

But in the formula you must use the components of $\vec{v}^*$.

STOP HERE on 10/15

② Gradient

Rewrite the formula for $D_{\vec{u}} f$:

$$D_{\vec{u}} f = f_x \cdot u_1 + f_2 \cdot u_2 \underset{\text{Sec. 12.3}}{=} \underbrace{\langle f_x, f_y \rangle}_{\text{gradient of } f} \bullet \underbrace{\langle u_1, u_2 \rangle}_{\vec{u}}$$

Gradient of $f(x,y)$ $\equiv \vec{\nabla} f(x,y) = \langle f_x, f_y \rangle$

"nabla"

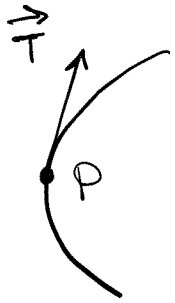
$$\equiv \vec{i} \cdot f_x + \vec{j} \cdot f_y.$$

Thus, $\boxed{D_{\vec{u}} f = \vec{\nabla} f \bullet \vec{u}}$.

Generalization for $f(x,y,z)$ etc. is straightforward: see Ex. 5 in book.

(3) Gradient, level curves, and the direction of steepest slope.

[a] Level curves



Let $f(x,y) = C$ be any level curve (of the given function).

($f(x,y)$ does not change along $f(x,y) = C$)

$\Downarrow$

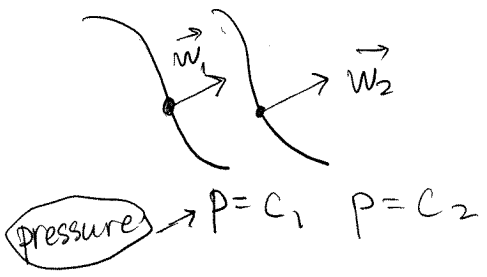
(At any point, $D_{\vec{T}} f = 0$, where $\vec{T}$ is the unit tangent vector to the given level curve at that point)

$\Downarrow$

$(\vec{\nabla} f \cdot \vec{T} = 0) \Rightarrow \boxed{\vec{\nabla} f \perp \vec{T}}$

The gradient is perpendicular to level curves at every point.

Example: Isobaric curves and wind

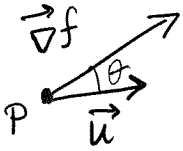


Wind $\vec{w}$ is proportional to $\vec{\nabla} p$

(because the cause of wind is change in pressure; wind blows from higher to lower pressure).

Thus $\vec{w} \perp$ (curve $p = \text{const}$).

6 Directions of steepest slope



Suppose one knows $\vec{\nabla} f$ at some point and asks:

Along what direction (= along what vector $\vec{u}$) will the slope of the surface $z = f(x,y)$ be the largest?

Slope along $\vec{u}$ $= D_{\vec{u}} f = \vec{\nabla} f \bullet \vec{u}$

$= |\vec{\nabla} f| \cdot \cancel{|\vec{u}|} \cdot \cos\theta$ ($|\vec{u}| = 1$)

$\max\limits_{\theta} D_{\vec{u}} f = \max\limits_{\theta} |\vec{\nabla} f| \cdot \cos\theta = |\vec{\nabla} f| \cdot 1$

(θ: among all θ; $|\vec{\nabla} f|$: does not depend on θ; $\cos\theta = 1$)

Conclusion 1: $\max\limits_{\theta} D_{\vec{u}} f = |\vec{\nabla} f|$ for $\theta = 0$

Function $f(x,y)$ increases the fastest along the direction of $\vec{\nabla} f$: $\vec{u} \uparrow\uparrow \vec{\nabla} f$. Equivalently,

If one climbs the surface (hill) $z = f(x,y)$, then:

- The steepest climb up (ascent) occurs along $\vec{\nabla} f$;
- The max upward slope is $|\vec{\nabla} f|$.

Similarly, $\min\limits_{\theta} D_{\vec{u}} f = \min\limits_{\theta} |\vec{\nabla} f| \cdot \cancel{|\vec{u}|} \cdot \cos\theta = |\vec{\nabla} f| \cdot (-1)$ ($\cos 180^\circ = -1$)

Conclusion 2: $\min\limits_{\theta} D_{\vec{u}} f = -|\vec{\nabla} f|$ for $\theta = 180^\circ$

Function $f(x,y)$ decreases the fastest along the direction opposite to that of $\vec{\nabla} f$: $\vec{u} \downarrow\uparrow \vec{\nabla} f$. Equivalently,

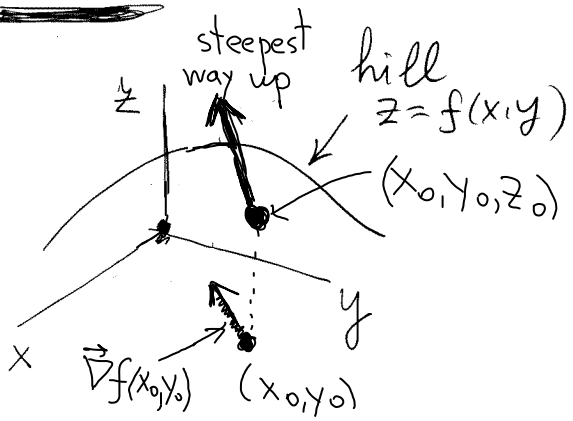
If one walks down the hill $z = f(x,y)$, then:

- The steepest way down (descent) occurs along $(-\vec{\nabla} f)$;
- The max downhill slope is $(-|\vec{\nabla} f|)$.

Yet another interpretation:

Suppose you are standing on a hill and look around. You will see the steepest slope up along the direction of $\vec{\nabla} f$, and you see the steepest slope down along the direction of $(-\vec{\nabla} f)$.

Note:



- $\vec{\nabla} f = \langle f_x, f_y \rangle$ lies in the xy-plane; so "the direction along $\vec{\nabla} f$" in the direction in the horizontal plane.

- When we say that the steepest slope up $= |\vec{\nabla} f|$, we mean that

$$\frac{\text{rise}}{\text{run}} = \frac{\text{vertical length of the step}}{\text{horizontal length of the step}} = |\vec{\nabla} f|;$$

see Ex. 2 below.

- Recall that in [a] we learned that $\vec{\nabla} f \perp$ (level curves).
- Along the level curves, the slope is 0 (the surface does not change its elevation (= is level) along the level curves).
- Recall Conclusions 1 & 2 above.

Conclusion 3: Direction of the steepest change of f (either ascent or descent) is $\perp$ direction of level curves.

See also Figs. 11–13 (end of Sec. 14.6) in the book.

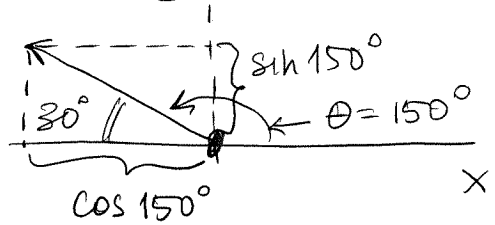
Ex. 2 Suppose you are walking on a hill with equation $z = x^2 + 5y^2 - 7$ and stop at a pt. $(8, 3, 102)$.

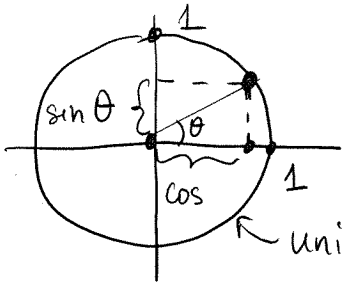
(a) If you go in the direction making angle $\theta = 150°$ with the x-axis, will you ascend or descend? At what rate?

(b) In what direction should you start walking to see the largest slope down? At what angle to the horizontal plane (the xy-plane) will you descend?

Sol'n: We first need to identify whether the problem asks about the directional derivative or the gradient.

(a) Here we know the direction and are asked about the slope in that direction. So this is the question about the directional derivative (follow Ex. 1).





1) $\vec{u} = \langle \cos 150°, \sin 150° \rangle$

$= \langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$.

2) $\vec{\nabla} f|_{(8,3)} = \langle 2x, 5 \cdot 2y \rangle|_{(8,3)}$

$= \langle 16, 30 \rangle$.

3) $D_{\vec{u}} f|_{(8,3)} = \langle 16, 30 \rangle \bullet \langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

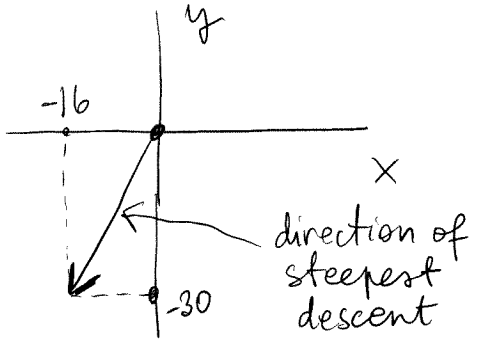
≈ 1.1

Since $D_{\vec{u}} f > 0$, you will ascend ($z \uparrow$).
Rate of ascent: use the units of your variables...

(b) This problem asks about the largest slope (down). Therefore, it is asking about the gradient (see Conclusion 2). We have found it in (a): $\vec{\nabla} f|_{(8,3)} = \langle 16, 30 \rangle$.

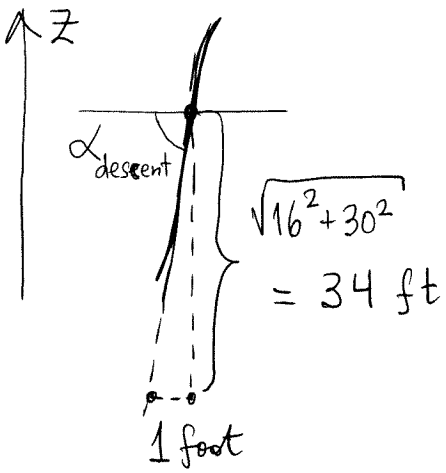
Therefore, the steepest descent occurs in the direction $-\vec{\nabla} f = \langle -16, -30 \rangle$.

Top view:



The slope in this direction $= -|\vec{\nabla} f|$.

Side view:



$$\tan \alpha_{descent} = \frac{34}{1} \Rightarrow \alpha \approx 88^\circ.$$

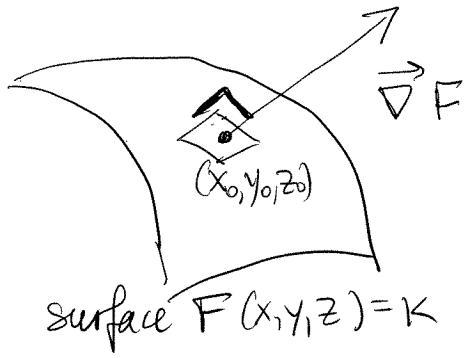
(4) Level surfaces and the gradient

Suppose we have $w = F(x,y,z)$ (say, temperature of a body; see end of Sec. 14.1). Its level surfaces $F(x,y,z) = k$ implicitly

define surfaces $z(x,y)$ where $w = k$
(think about surfaces of equal temperature in a body; see end of Sec. 14.1 / Notes).

Using the analogy with topic 3[a] about the level curves $f(x,y)=k$, we can state:

$\underbrace{\vec{\nabla} F(x,y,z)}_{\langle F_x, F_y, F_z \rangle} \perp \left(\begin{array}{c} \text{level surfaces} \\ F(x,y,z) = k \end{array} \right)$ at every point of the surface.



$\vec{\nabla} F \Big|_{(x_0,y_0,z_0)} = \langle F_x, F_y, F_z \rangle @ (x_0,y_0,z_0)$

Now, since

"$\vec{\nabla} F \perp$ surface"

means

"$\vec{\nabla} F \perp$ tangent plane",

then the equation of the tangent plane through (x_0,y_0,z_0) must be (see Sec. 12.5 B):

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0 \quad (\star)$$

How come it looks different from the eq. of the tan. plane to the surface $z = f(x,y)$ from Sec. 14.4 ?

Oh, wait!

$z = f(x,y) \Rightarrow \underbrace{f(x,y) - z}_{F(x,y,z)} = 0$

$F_x(x,y,z) = (f(x,y) - z)_x = f_x(x,y) - \frac{\partial z}{\partial x} = f_x(x,y)$

$F_y(x,y,z) = (f(x,y) - z)_y = f_y(x,y) - \frac{\partial z}{\partial y} = f_y(x,y)$

$F_z(x,y,z) = \frac{\partial f(x,y)}{\partial z} - 1 = -1.$

Substitute these into eq. (★):

$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + (-1)(z - f(x_0,y_0)) = 0$

$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f(x_0,y_0) = z,$

which is the same as the eq. of the tan. plane from Sec. 14.4.

MUST SEE Ex. 8 in book for:

- (★) with numbers;
- The eq. of the normal line to a surface.