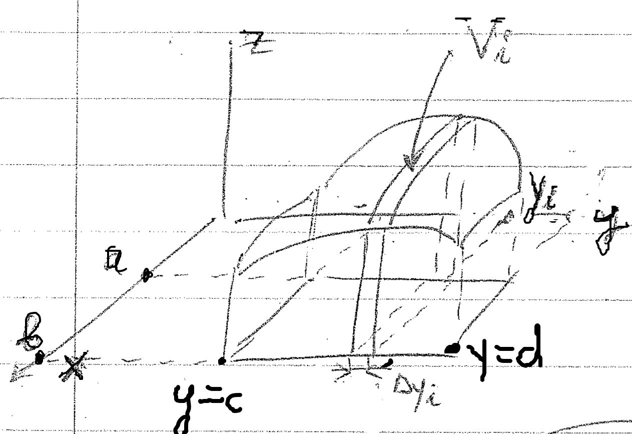
Sec. 15.1-B: Double integral as an iterated integral

1 Main result

We need a method to evaluate double integrals.
Here we do it under 2 assumptions:

1) The domain R is a rectangle whose sides are $\parallel$ to the x- and y-axes;
2) $f(x,y)$ is continuous.

$\boxed{R=[a,b]\times[c,d]}$



We start from

$$\iint_R f(x,y)\,dA = V, \quad (1)$$

$$V=\sum_i V_i, \text{ where the slice } V_i \text{ is shown.} \quad (2)$$

Now, V_i = base $\cdot$ height = (face area) $\cdot\,\Delta y_i$ (3)

face area = Area under $f(x,y_i)$ for $x\in[a,b]$

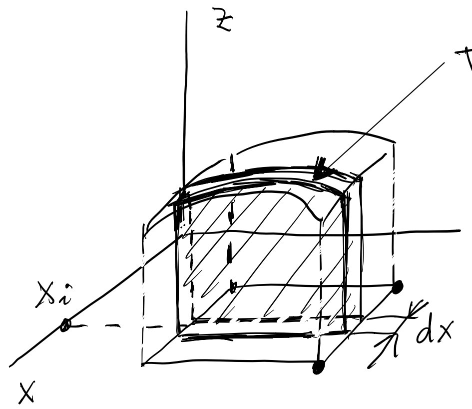
$$= \int_a^b f(x,y_i)\,dx \quad (4)$$

Put (1) – (4) together:

$$\iint_R f(x,y)\,dA = V = \sum_i V_i \approx \sum_i \left(\int_a^b f(x,y_i)\,dx\right)\Delta y_i,$$

$$\Rightarrow \iint_{[a,b]\times[c,d]} f(x,y)\,dA = \int_c^d \left(\int_a^b f(x,y)\,dx\right)dy$$

However, by the same token, we could have sliced the volume along the y-axis:



$$\iint_{[a,b]\times[c,d]} f(x,y)\, dA = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$$

<u>**Fubini's Thm**</u>: Let $R = [a,b]\times[c,d]$ and $f(x,y)$ is <u>continuous</u> on R. Then:

$$\iint_R f(x,y)\, dA = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$$

- **In practice, one performs the easier integration first. (I.e., use the more convenient order of integration.)**
- **When integrating over x, treat y as const, and vice versa.**

<u>Ex. 1</u> Find the volume bounded by the surface $z = y^3 \sin(xy^2)$ and the xy-plane over the rectangle $R = [0,\pi]\times[0,1]$ (i.e., $0 \le x \le \pi$, $0 \le y \le 1$).

<u>Sol'n</u>: **0) In almost all problems in Chap. 15, we (= you!) need to make a sketch <u>before</u> <u>doing anything else</u>.** However, in <u>this</u> problem, we <u>don't</u>.

These <u>exceptional cases</u> where you do <u>not</u> need to sketch, will be announced. **By default, you need to sketch!**

1) Integration over y would require a u-substitution $u = y^2$ followed by integration by parts. Possible, but not easy.

2) So we try to integrate over x first.

$$\iint_R y^3 \sin(xy^2)\, dA = \int_0^1 \Big(\underbrace{\int_0^{\pi} y^3 \sin(xy^2)\, dx \Big)}_{\text{Inner } \int} dy$$

$$\text{Inner } \int = \int_{x=0}^{x=\pi} y^3 \sin u \cdot \frac{du}{y^2}$$

$u = xy^2$ ← function of x only!

$$= \int_{x=0}^{x=\pi} y \cdot \sin u \cdot du$$

y ← const!

$du = y^2 \cdot dx$ ← const, when taking x-derivative

$$= y \cdot \int_{x=0}^{x=\pi} \sin u \cdot du = y \cdot (-\cos u) \Big|_{x=0}^{x=\pi}$$

$dx = du / y^2$

$$= y \cdot (-\cos(xy^2)) \Big|_{x=0}^{x=\pi}$$

$$= -y \left(\cos(\pi y^2) - \underbrace{\cos(0 \cdot y^2)}_{\to 1} \right) = y(1 - \cos(\pi y^2)).$$

Note: We explicitly indicated that the limits were for x, not u, even though we integrated over u. By default, when we integrate over u, the limits would be for u.

Later we will practice u-sub for limits.

3) $$\text{Outer} \int = \int_0^1 y(1 - \cos(\pi y^2))\, dy = \underbrace{\int_0^1 y\, dy}_{I_1} - \underbrace{\int_0^1 y \cos(\pi y^2)\, dy}_{I_2}$$

$I_1 = \int_0^1 y\,dy = \frac{y^2}{2}\Big|_0^1 = \frac{1}{2}$

$I_2 = \int_0^1 \underline{y} \cos(\pi y^2)\,\underline{dy} =$

$u = \pi y^2$

$du = \pi \cdot 2y\,dy$

$\boxed{\left(du = \frac{du}{dy} \cdot dy\right) \quad \text{do not drop!}}$

$y\,dy = du/(2\pi)$

$= \int_{y=0}^{y=1} \cos u \,\frac{du}{2\pi} = \frac{\sin u}{2\pi}\Big|_{y=0}^{y=1}$

$= \frac{\sin(\pi y^2)}{2\pi}\Big|_{y=0}^{y=1} = \frac{\cancel{\sin(\pi\cdot 1)}^{0} - \cancel{\sin 0}^{0}}{2\pi} = 0.$

Combine: $\iint_R f(x,y)\,dA = \text{Outer} \int = \underbrace{I_1}_{1/2} - \underbrace{I_2}_{0} = 1/2.$

See also Ex. 5, 6, 7 in book.

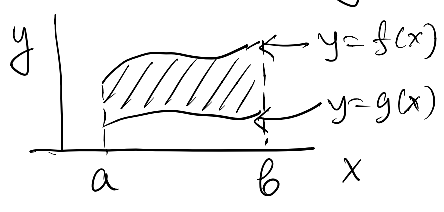
② Special case: If $f(x,y) = f_1(x) f_2(y)$

$\Rightarrow \int_a^b \int_c^d f_1(x) f_2(y)\,dy\,dx = \int_a^b f_1(x)\,dx \cdot \int_c^d f_2(y)\,dy.$

See Ex. 8 in book.

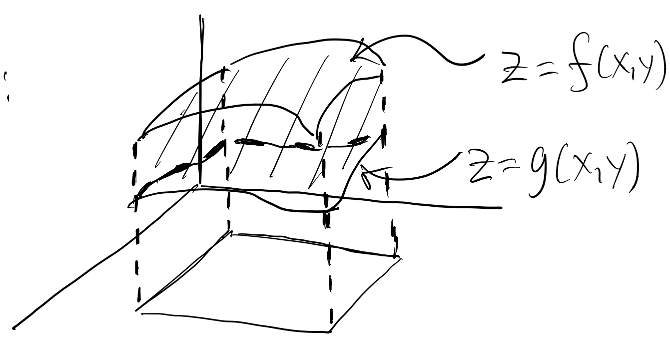
③ Volume bounded by $z = f(x,y)$ & $z = g(x,y)$

Calc. I:



Area $= \int_a^b (f(x) - g(x))\,dx.$

Calc. III:



Volume $= \iint_R (f(x,y) - g(x,y))\,dA$