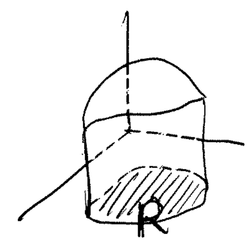
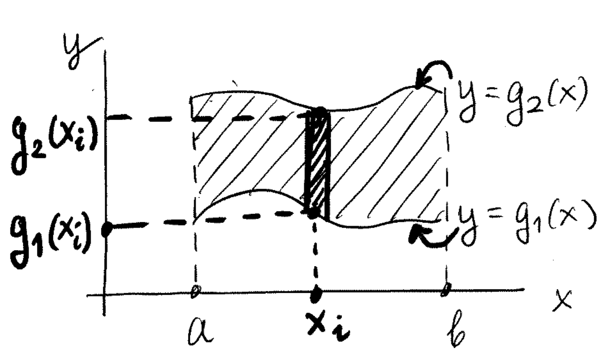
Sec. 15.2 Double integrals over non-rectangular regions

① Main result

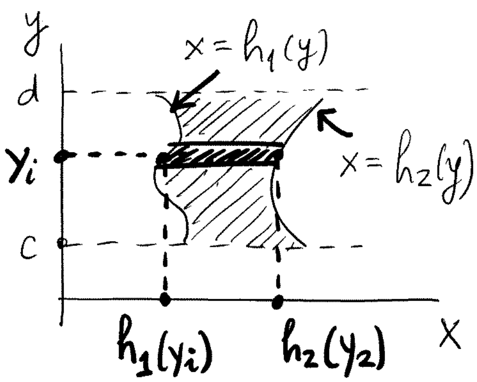
We will consider the situation when the integration region R in $\iint_R f(x,y)\,dA$ is of one of these Types:



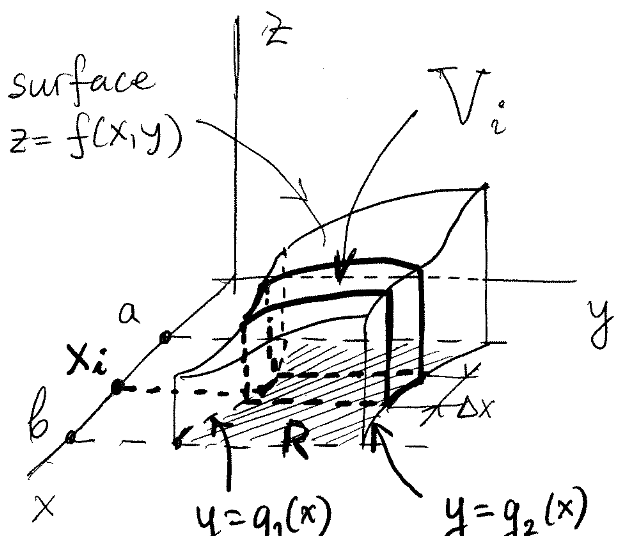
Type I



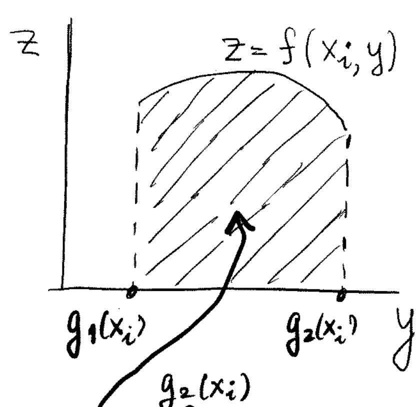
Type II



- Let us first consider how $\iint_R f\,dA$ can be written as an iterated integral when R is Type I:



View of the slice's face:



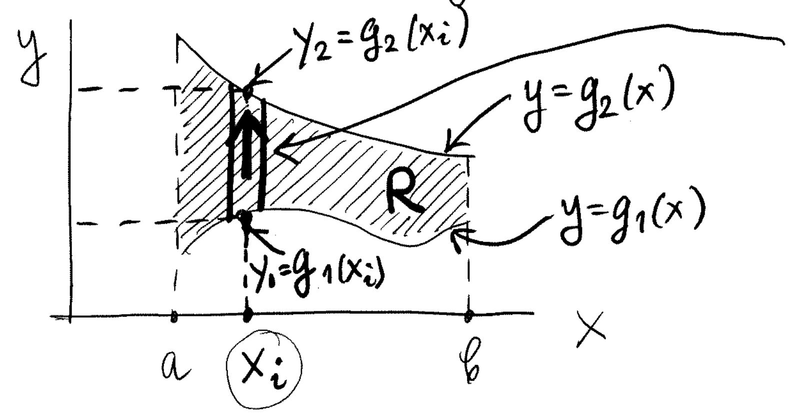
$$\text{Area} = \int_{g_1(x_i)}^{g_2(x_i)} f(x_i, y)\,dy$$

$$\boxed{V = \sum_i V_i \approx \sum_i (\text{Area of face})_i \cdot \begin{pmatrix}\text{thickness}\\ \text{of slice}\end{pmatrix}}$$

Type I

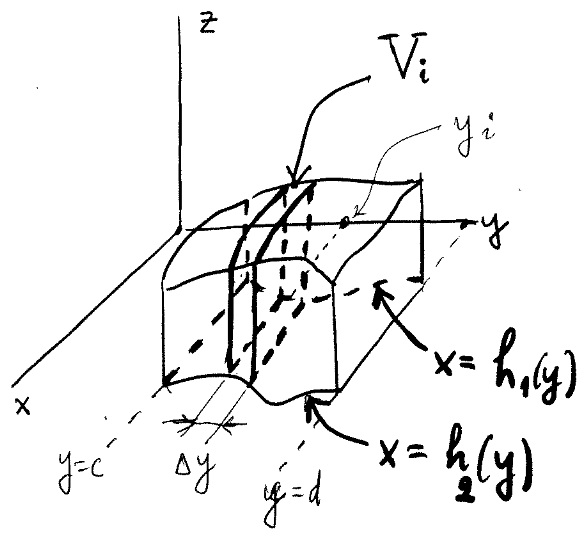
$$= \sum_i \underbrace{\left(\int_{g_1(x_i)}^{g_2(x_i)} f(x_i, y)\,dy \right)}_{F(x_i)} \Delta x \xrightarrow[\Delta x \to 0]{} \boxed{\int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx}$$

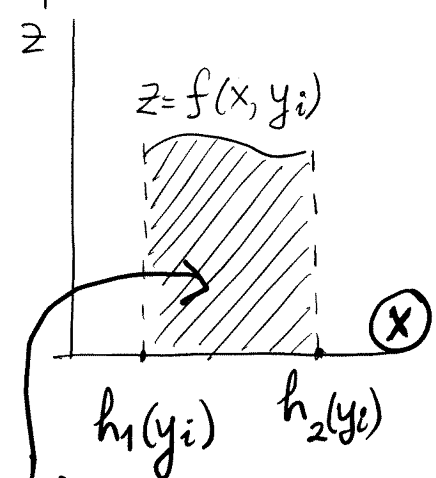
Thus, we first integrated $f(x,y)$ **along the vertical strip** and then added up the results for all such strips for $a \le x_i \le b$.



• We can similarly consider the case when R is Type II.

View of the slice's face:





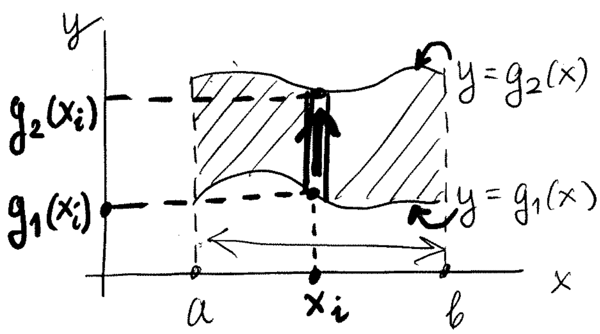
$$\text{Area} = \int_{h_1(y_i)}^{h_2(y_i)} f(x, y_i)\,dx$$

Then similarly to the calculation on p. 20-2:

$$V = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy \quad \text{(Type II)}$$

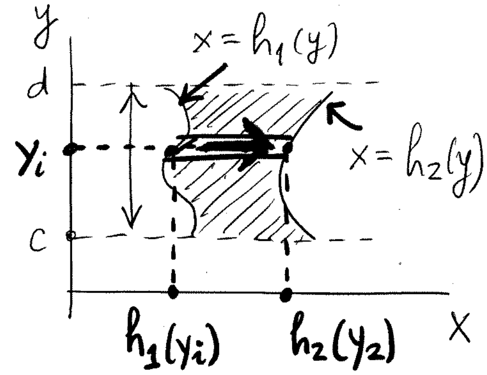
To summarize:

Type I



$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx$$

Type II



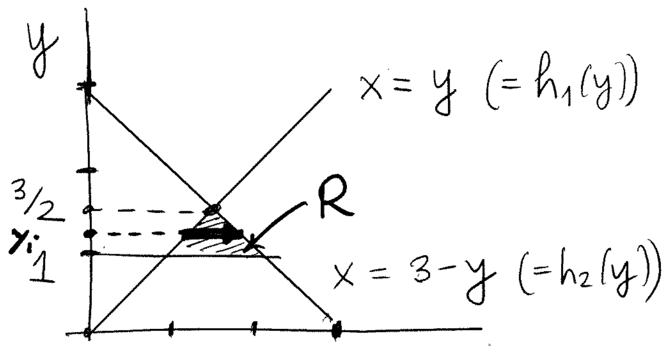
$$\iint_R f(x,y)\,dA = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy$$

Mnemonic rules:

- In the $\int \dots dx$, the limits cannot depend on x. They can depend on y or be constant.
- In the $\int \dots dy$, the limits cannot depend on y. They can depend on x or be constant.
- The limits of the outer $\int$ can NEVER depend on x or y. They can only be constant!

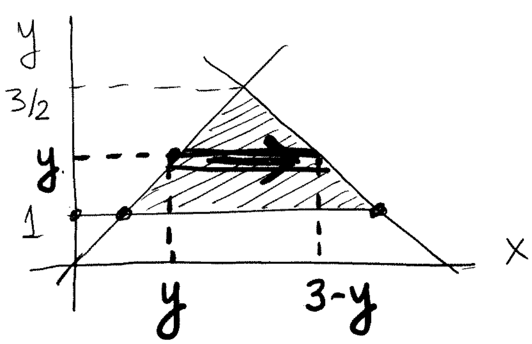
<u>Ex. 1</u> Find $I = \int_1^{3/2} \left(\int_y^{3-y} (x+y)\,dx \right) dy$. ← Type II

<u>Sol'n:</u> 0) In Chap. 15, you will need to <u>sketch</u> in almost every problem. Here, one does not have to do it because the limits of integration are already determined. However, we will sketch just to practice.



<u>Q:</u> What is wrong about this sketch?

<u>A:</u> It is <u>better</u> to have a sketch that is <u>not to scale</u> <u>but shows all details clearly</u> then one that is to scale but crams details in a small area.



$$\text{Inner} \int = \int_y^{3-y} (x+y)\,dx = \underbrace{\int_y^{3-y} x\,dx}_{I_1} + \underbrace{\int_y^{3-y} y\,dx}_{I_2}.$$

$$I_1 = \frac{x^2}{2}\Big|_y^{3-y} = \frac{1}{2}\left((3-y)^2 - y^2\right) = \frac{9}{2} - 3y$$

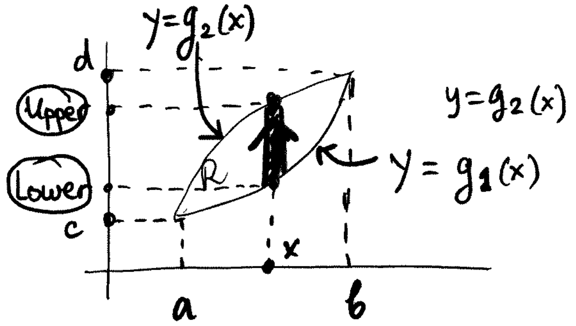
$$I_2 = \int_y^{3-y} \underset{\substack{\uparrow \text{const,} \\ \text{when integrating} \\ \text{over } x}}{y}\,dx = y\int_y^{3-y} 1\cdot dx = y\cdot\left(x\Big|_y^{3-y}\right) = y(3-y-y) = 3y - 2y^2.$$

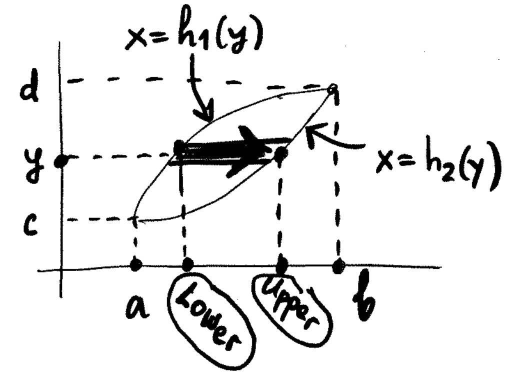
$$I = \int_1^{3/2} (I_1 + I_2)\,dy = \int_1^{3/2} \left(\frac{9}{2} - \cancel{3y} + \cancel{3y} - 2y^2\right) dy$$

$$= \left(\frac{9}{2}y - \frac{2}{3}y^3\right)\Big|_1^{3/2} = 17/12.$$

See also Ex. 1, 3 in book (similar).

Sometimes, region R is both Type I & Type II:





As Type I:

$$\int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx$$

As Type II:

$$\int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy$$

Then one chooses the integration order that is the more convenient (or, sometimes, the only one possible).

Ex. 2 Practice reversal of the integration order.

Express the given $\iint$ as an equivalent $\iint$ with the order of integration reversed.

$$I = \int_0^2 \int_{y^2}^4 f(x,y)\,dx\,dy$$

Sol'n:

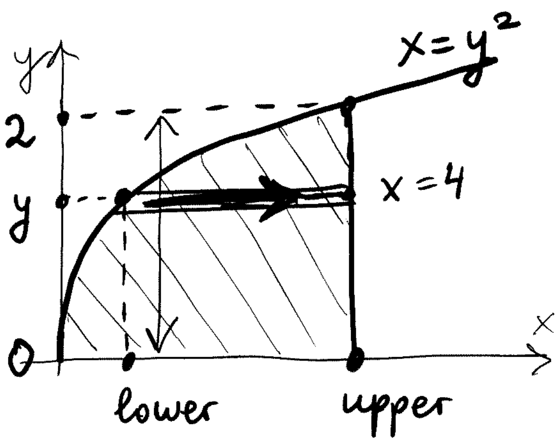
0) SKETCH!

- The $\iint$ is set up as Type II. Comparing:

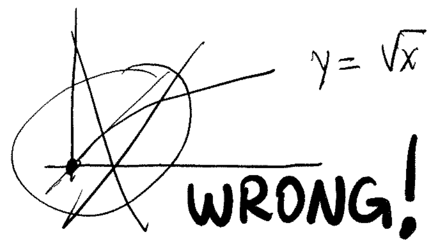
$$\int_0^2 \left(\int_{y^2}^4 f(x,y)\,dx \right) dy \equiv \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy$$

$$\Rightarrow \quad h_1(y) = y^2 \equiv x_{lower}$$
$$h_2(y) = 4 \equiv x_{upper}$$
$$c = 0 \equiv y_{min}$$
$$d = 2 \equiv y_{max}.$$

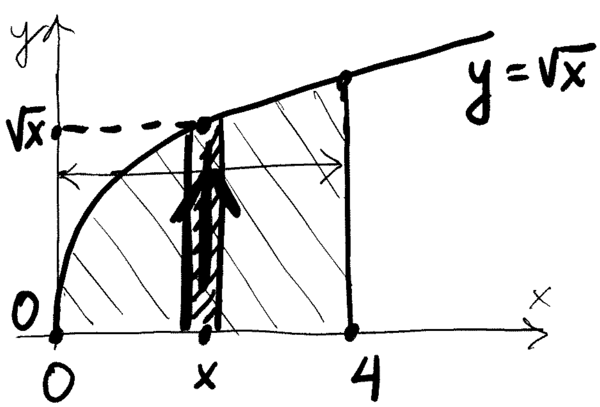
- The actual sketch, using h_1, h_2, c, d:



- **Note:** $y = \sqrt{x}$ has the vertical tangent at $x = 0$.



1) Look at the sketch and re-cast the region as Type I.



Now:

$$\underset{y_{low}}{0} \le y \le \underset{y_{upp}}{\sqrt{x}}$$

$$\underset{x_{min}}{0} \le x \le \underset{x_{max}}{4}$$

Thus: $$I = \int_0^4 \left(\int_0^{\sqrt{x}} f(x,y)\, dy \right) dx.$$

! Never, ever try to manipulate "old" limits to obtain "new" ones.

Instead:

1. Make a sketch using the given limits.
2. Use the sketch (NOT the old limits!) to obtain new limits.

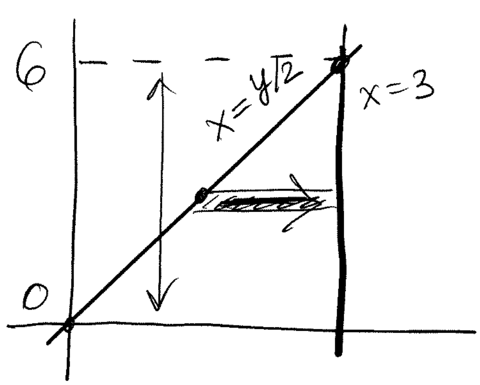
Ex. 3. Evaluate $I = \int_0^6 \left(\int_{y/2}^{3} \sin(x^2)\,dx \right) dy$.

Sol'n: 1) Cannot do $\int \sin(x^2)\,dx$ — the anti-derivative of $\sin(x^2)$ is not an elementary function! So, switch the integration order.

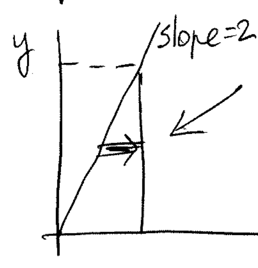
2) SKETCH!

- The region is set up as Type II.

$y/2 \le x \le 3$; $0 \le y \le 6$

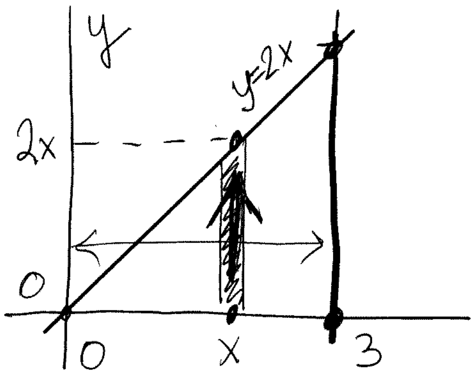


Recall the comment on p. 20-4: Sacrifice "to scale" for visibility of details:



Not terrible, but not as clear as the other one.

- Re-sketch and rewrite as Type I



$0 \le y \le 2x$, with $0 = y_{low}$ and $2x = y_{upp}$

$0 \le x \le 3$, with $0 = x_{min}$ and $3 = x_{max}$

$$I = \int_0^3 \left(\int_0^{2x} \sin(x^2)\, dy \right) dx .$$

3) Compute.

$$\text{Inner} \int = \int_0^{2x} \underbrace{\sin(x^2)}_{\text{= const when integrating over } y!} dy = \sin(x^2) \int_0^{2x} dy = \sin(x^2)\cdot 2x.$$

$$\text{Outer} \int = \int_0^3 \sin(x^2)\cdot 2x \cdot dx$$

$$= \int_0^9 \sin u \cdot du$$

limits for u, not x

$$= -\cos u \Big|_0^9$$

$$= -(\cos 9 - \cos 0)$$

$$= 1 - \cos 9$$

1) $u = x^2$

$du = (x^2)' dx$
$\quad = 2x\, dx$

2) Change limits!

$u_{low} = 0^2 = 0$

$u_{upp} = 3^2 = 9$

Answer: $\iint = 1 - \cos 9.$ ✓

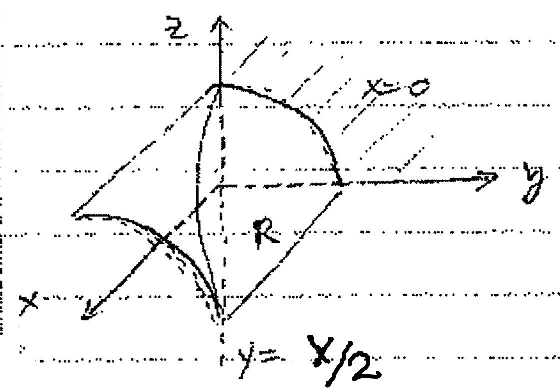
See also Ex. 2, 5 in book.

When finding the volume of a solid, one may need to <u>determine</u> the int. limits. See the next example.

Ex. 4 Find the volume of a solid bounded by the cylinder $y^2+z^2=4$ and the planes $y = x/2$, $x=0$, $z=0$ in the first octant.

Sol'n:

1) Sketch the solid

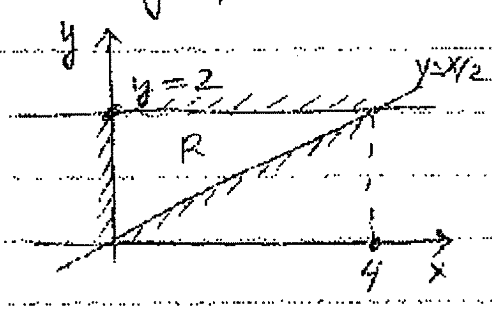


- $y^2+z^2=4$ – circular cylinder of radius 2, axis = x-axis

$Z = \oplus\sqrt{4-y^2}$ ← the part above the xy-plane

- $z=0$
- $x=0$
- $y = x/2$

2) Find the boundaries of region R in the xy-plane



(a) If the top surface is $z=f(x,y)$, set $z=f(x,y)=0$ ← bottom (in this problem)
(intersection w/ xy-plane)

$\sqrt{4-y^2} = 0 \Rightarrow$

$y^2=4 \Rightarrow y = +2$ or $\cancel{-2}$ ← need 1st octant only

(b) the other boundaries are given: $y = x/2$; $x=0$

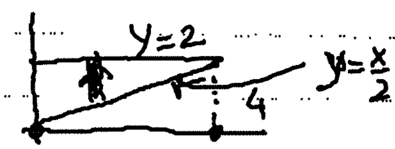
- This is both Type I and Type II region, so may need to try both.

3) $$V = \iint_R \left(z_{top}(x,y) - z_{bot}(x,y) \right) dA$$

$\uparrow$ $\sqrt{4 - y^2}$ $\quad$ $\uparrow$ 0

$$= \iint_R \sqrt{4-y^2}\, dA.$$

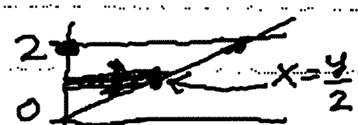
(a) Try Type I setup:



$$V = \int_{x=0}^{x=4} \left(\int_{y=x/2}^{y=2} \sqrt{4-y^2}\, dy \right) dx.$$

Doable, but requires trig. sub (Calc. II).

So try Type II -setup:



(b) $$V = \int_{y=0}^{y=2} \left(\int_0^{2y} \sqrt{4-y^2}\, dx \right) dy$$

$\uparrow$ const. when integrating over x

$$= \int_0^2 \sqrt{4-y^2} \cdot \left(x \Big|_0^{2y} \right) dy = \int_0^2 \sqrt{4-y^2} \cdot \underbrace{2y\,dy}_{-du}$$

$$= \int_4^0 \sqrt{u}\,(-du) = \int_0^4 u^{1/2}\,du = \frac{u^{3/2}}{3/2}\Big|_0^4 = \frac{16}{3}.$$

$$\boxed{\begin{aligned} u &= 4 - y^2 \\ du &= (4-y^2)'dy \\ &= -2y\,dy \\ u_{low} &= 4 - 0^2 = 4 \\ u_{upp} &= 4 - 2^2 = 0 \end{aligned}}$$

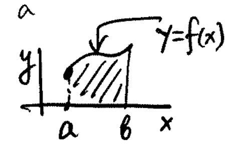
(2) Properties of double integrals (especially $\boxed{6}$, $\boxed{11}$)

Read the last 1.5 page of Sec. 15.2, including Ex. 6.

— they are straightforward and same as for the single integral.

3) Double integrals using symmetry

Idea: The inner $\int$ in $\iint_R$ can be viewed as **area** under some curve ($\int_a^b f(x)dx$), and this sometimes can be used to find it w/o a calculation.

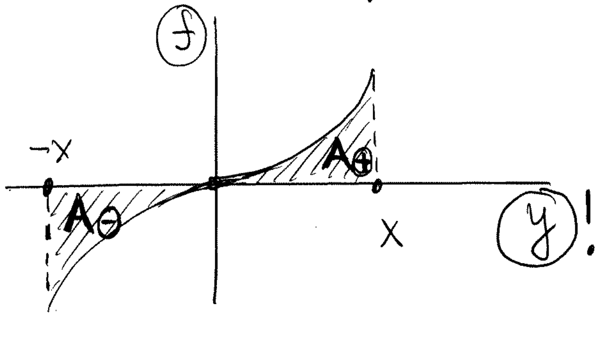


(This was considered in Calc. I: end of Sec. 5.5.)

Ex. 5 Find $\int_0^{\pi}\Big(\int_{-x}^{x} \underbrace{y^3\sin(xy^2)}_{f(x,y)}\,dy\Big)\,dx$.

Note that $f(x,y)$ is the same as in Ex. 1 of Sec. 15.1-B (but the limits of $\iint$ were different). There, we had to switch the integration order ($dy\,dx \to dx\,dy$) so as to avoid difficult integration. Here, we will use the special form of the inner limits to compute the integral w/o a calculation.

Sol'n: 1) Sketch the function $f(x,y)$ within the inner integration limits.



$f(x,y) = y^3 \cdot \sin(xy^2)$ ← keep as const

This function of y is odd: $f(x,-y) = f(x,y)$.

Indeed:

$$f(x,-y) = \underbrace{(-y)^3}\cdot \sin(x\underbrace{(-y)^2})$$

$$= -y^3 \cdot \sin(xy^2) = -f(x,y). \checkmark$$

Therefore, in the figure above:

Area under f from $y=0$ to $y=x$

$=$

Area above f from $y=-x$ to $y=0$

$=$

$-$(Area "under" f from $y=-x$ to $y=0$).

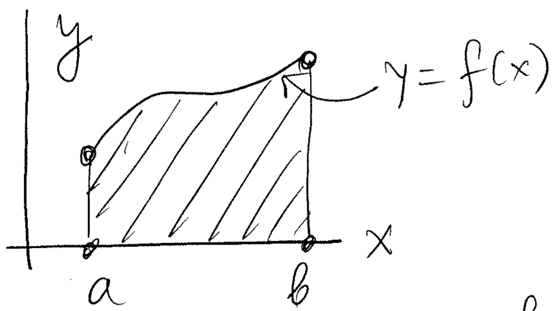
Thus, total area "under" f from $-x$ to x
$= 0$! (The "!" expresses the excitement about this answer; it does not mean "zero factorial". ☺)

- Note: In some HW problems, you'll find a similar situation.
  But in others, you will be asked to recognize the area as that of a familiar shape and thus find it w/o a calculation.

(4) Double integral as area

In Sec. 15.1A we introduced a $\iint_R$ as some volume. Now we will show that some $\iint_R$'s can also be viewed as area.

Analogy from Calc. I



We know that

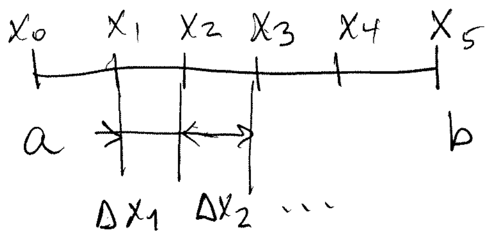
$$\int_a^b f(x)\,dx = \text{Area under } y=f(x) \text{ above } [a,b].$$

But what about $\int_a^b dx$?

By direct calculation: $\int_a^b 1\cdot dx = x\Big|_a^b = (b-a)$

$= \text{length of } [a,b].$

By original definition:

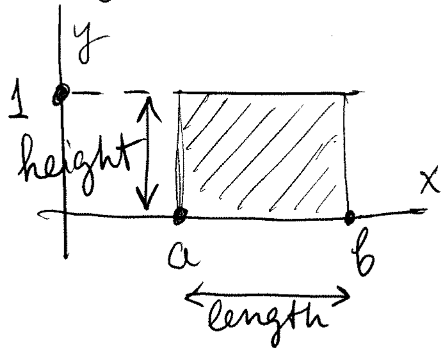


$$\int_a^b dx = \lim_{\Delta x \to 0} \sum_i \Delta x_i$$

$= (\Delta x_0 + \Delta x_1 + \dots \Delta x_n)$

$= \text{length of } [a,b].$

"By area":



$$\int_a^b 1\cdot dx = \text{Area under } y=1 \text{ over } [a,b]$$

$= \text{length} \cdot \cancel{\text{height}}^{\,1}$

$= \text{length (of } [a,b]).$

Thus, we will generalize:

Calc. I

$\int_a^b f(x)\,dx$ $\downarrow$ (Area under $y=f(x)$ above $[a,b]$)

$\int_a^b 1\cdot dx$ $\downarrow$ Length of $[a,b]$.

Calc. III

$\iint_R f(x,y)\,dA$ $\downarrow$ Volume under $z=f(x,y)$ above R

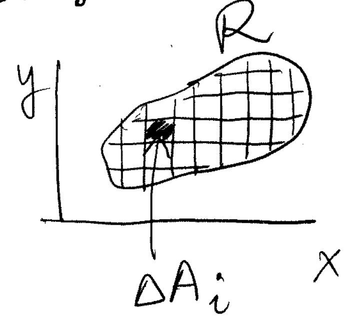
$$\boxed{\iint_R 1\cdot dA \downarrow \text{Area of } R.}$$

- Interpretation by definition:

$$\iint_R dA = \lim_{\Delta A\to 0} \sum_i \Delta A_i$$

$= \text{Sum of areas } \Delta A_i$

$= \text{Total area of } R.$



- Interpretation "by volume":

$\iint_R 1\cdot dA =$ Volume under $z=1$ above R

$= \begin{pmatrix}\text{Area of}\\ \text{base}\end{pmatrix} \cdot \cancel{\text{height}} \rightarrow 1$

$=$ Area of R.