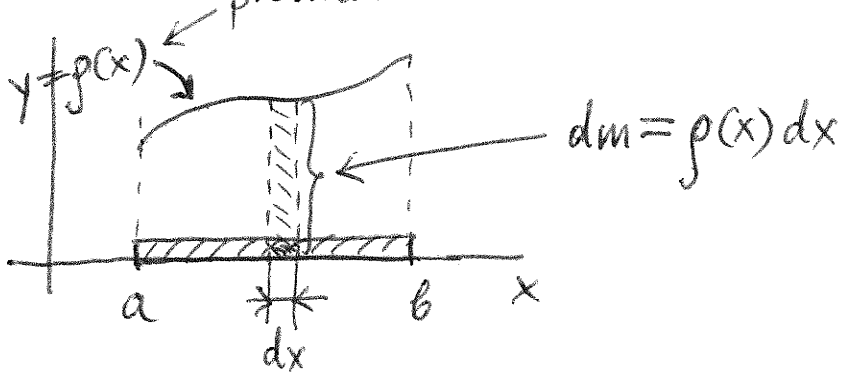
Sec. 15.4

Some applications of double integrals.

(1) Mass

Calc. I



Consider a non-uniform beam with mass density $\rho(x)$.

$$M = \sum_i M_i \approx \sum_i \overbrace{\rho(x_i)\,\Delta x}^{M_i}$$

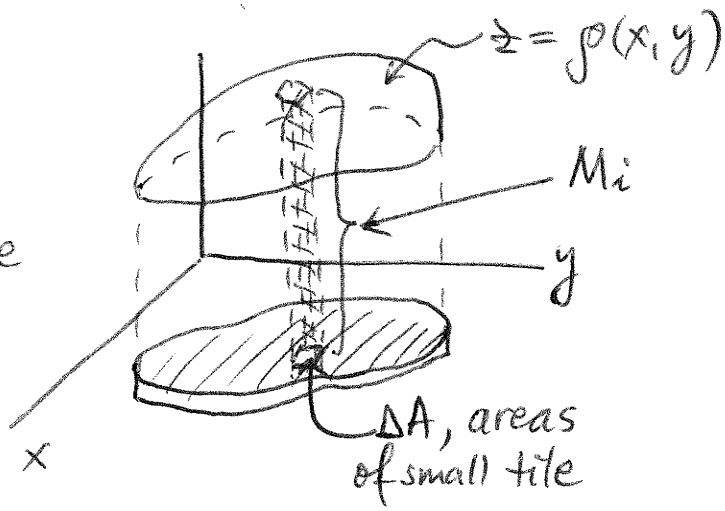
M ↑ total mass

As $\Delta x \to 0$,

$$\boxed{M = \int_a^b \rho(x)\,dx}$$

Calc. III

Non-uniform plate (lamina) with mass density $\rho(x,y)$.



$$M = \sum_i M_i = \sum_i \rho(x_i, y_i)\,\Delta A \Rightarrow \iint_R \rho(x,y)\,dA.$$

Thus:

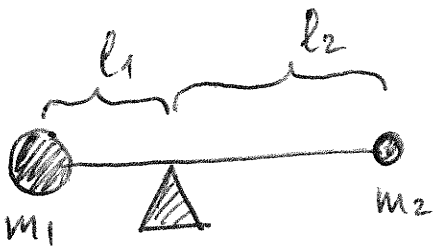
$$\boxed{M = \iint_R \rho(x,y)\,dA}$$

Note: Recall the special case $\rho(x,y) = 1$: then

$$\iint_R 1 \cdot dA = \left(\begin{matrix}\text{Area} \\ \text{of } R\end{matrix}\right).$$

(2) Center of mass

[a] Motivation.



Consider a lever made of two point-like masses m_1, m_2 connected by a massless rod (as shown).

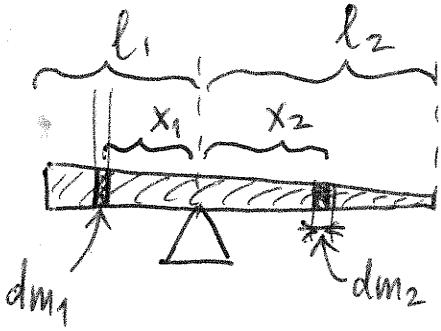
The lever is in balance when:

$$\boxed{m_1 l_1 = m_2 l_2} \qquad (*)$$

("law of the lever", corollary of equation for mechanical torque).

[b] Calc. I

Now consider the problem of balancing a non-uniform beam:



This extended lever will be balanced if every dm_1 on the left of the fulcrum (= the support) is balanced by <u>some</u> dm_2 on the right:

$$dm_1 \cdot x_1 = dm_2 \cdot x_2 .$$

Integrating each side, we get:

$$\int_0^{\ell_1} x_1\, dm_1 = \int_0^{\ell_2} x_2\, dm_2 , \quad \text{or}$$

$$\boxed{\int_0^{\ell_1} x_1\, \rho(x_1)\, dx_1 = \int_0^{\ell_2} x_2\, \rho_2(x_2)\, dx_2 .} \quad (\star\star)$$

Balance condition of an extended lever.

We will <u>not</u> pursue this specific problem, but it <u>motivates</u> the concept of <u>center of mass</u>:

$$\int_0^{\ell} x\, \rho(x)\, dx \equiv M \cdot \bar{x}$$

($\equiv$: "can be viewed as"; $\bar{x}$: some "effective" length where all M is lumped; M: total mass)

Then the condition $(\star\star)$ takes on the form:

$$M_1 \cdot \bar{x}_1 = M_2 \cdot \bar{x}_2 ,$$

which is the same form as $(\star)$ in [a].

Returning to

$$\int_0^\ell x\,\rho(x)\,dx = M \cdot \bar{x}$$

M: total mass

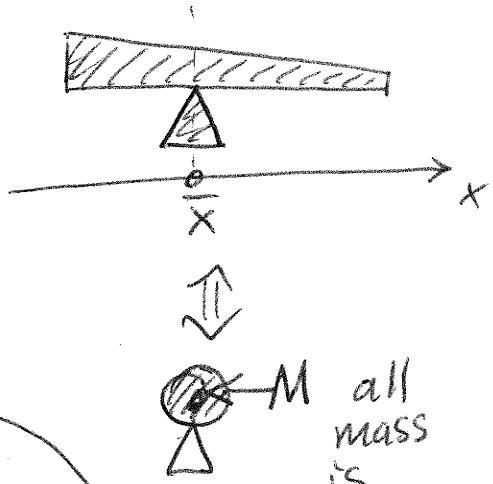
$\bar{x}$: "effective" location of lumped mass

$$\Rightarrow \boxed{\bar{x} = \frac{1}{M}\int_0^\ell x\,\rho(x)\,dx}$$

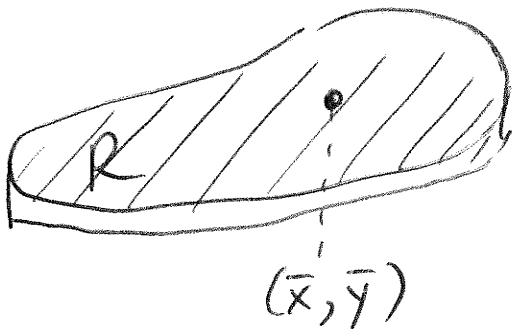
$\bar{x}$: center of mass (name)

(Recall that $M = \int_0^\ell \rho(x)\,dx$.)

One can show that if one places the support precisely under $\bar{x}$, then the beam will be in an equilibrium.



© Calc. III



Non-uniform plate R

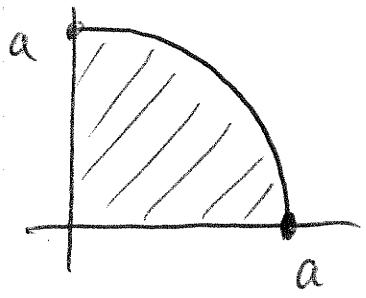
Its center of mass (CoM) is at $(\bar{x}, \bar{y})$:

$$\bar{x} = \frac{1}{M} \iint_R x\, \rho(x,y)\, dA$$

$$\bar{y} = \frac{1}{M} \iint_R y\, \rho(x,y)\, dA.$$

If the support is placed precisely under the CoM, the lamina will be in an equilibrium.

Ex. 1



Find the center of mass of the quarter of a circular plate shown in the figure, whose density is proportional to the distance from the origin.

Sol'n: $\bar{x} = \frac{1}{M} \iint_R x\, \rho(x,y)\, dA$,

$\Rightarrow$ need M first.

1) $M = \iint_R \rho(x,y)\, dA \underset{\text{use polar}}{=} \int_0^{\pi/2} \left(\int_0^a \underbrace{k \cdot r}_{\rho} \cdot r\, dr \right) d\theta$ (const k)

$$= \int_0^{\pi/2} k \frac{a^3}{3}\, d\theta = \frac{k}{3} a^3 \cdot \frac{\pi}{2}.$$

2) We'll do $\bar{x}$ in polar coord's and then $\bar{y}$ in Cartesian, for comparison.

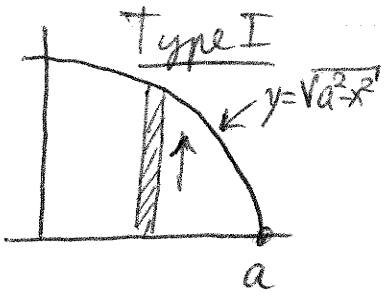
$$\bar{x} = \frac{1}{M}\iint_R x\,\rho(x,y)\,dA = \frac{1}{M}\int_0^{\pi/2}\left(\int_0^a r\cos\theta \cdot k\cdot r\cdot r\,dr\right)d\theta$$

$$= \frac{1}{M}\int_0^{\pi/2}\cos\theta\cdot\underbrace{\left(\int_0^a kr^3\,dr\right)}_{\frac{ka^4}{4}}d\theta = \frac{1}{M}\cdot\frac{ka^4}{4}\int_0^{\pi/2}\cos\theta\,d\theta$$

$$= \frac{1}{M}\cdot\frac{ka^4}{4}\cdot\underbrace{\sin\theta\Big|_0^{\pi/2}}_{1-0} = \frac{ka^4}{4}\Big/\underbrace{\frac{ka^3\cdot\pi}{6}}_{M}$$

$$= a\cdot\frac{3}{2\pi} \approx \frac{a}{2}$$

3) $$\bar{y} = \frac{1}{M}\iint_R y\,\rho(x,y)\,dx\,dy$$



$$= \frac{1}{M}\int_0^a\left(\int_0^{\sqrt{a^2-x^2}} y\cdot k\cdot\sqrt{x^2+y^2}\,dy\right)dx$$

$$= \frac{1}{M}\int_0^a\left(\int_{x^2}^{a^2} k\sqrt{u}\,\frac{du}{2}\right)dx$$

$$= \frac{k}{2M}\int_0^a \frac{u^{3/2}}{3/2}\Big|_{x^2}^{a^2} = \frac{k}{M\cdot 3}\int_0^a (a^3-x^3)\,dx$$

$$= \frac{k}{3M}\left(a^4-\frac{a^4}{4}\right) = \frac{ka^4}{4M} = \bar{y} \leftarrow \text{as expected.}$$

$u = x^2+y^2$

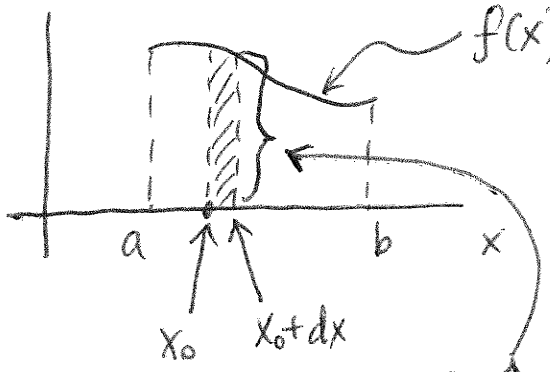
$du = 2y\,dy$

$u(0) = x^2+0 = x^2$

$u(\sqrt{a^2-x^2}) = x^2+a^2-x^2 = a^2.$

(3) Probability

Calc. I



$f(x)$ = probability density function (PDF)

Meaning of PDF:

$f(x_0)\,dx$ = probability that "x" is found inside $[x_0, x_0+dx]$

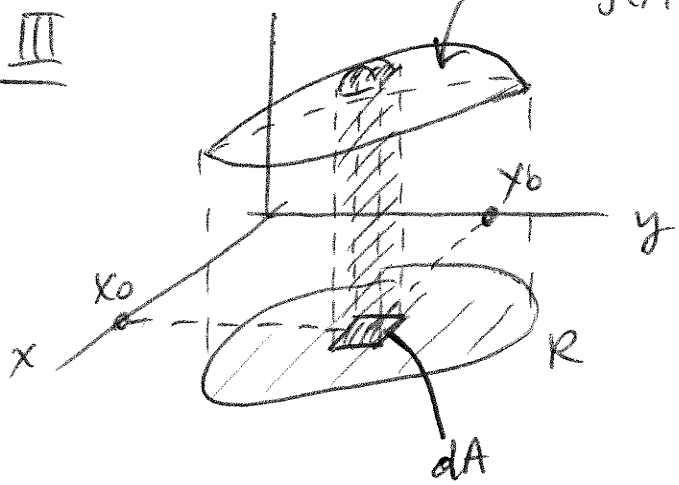
Then

$P(a \le x \le b) = \int_a^b f(x_0)\,dx$

probability that x is in $[a,b]$

sum of all these probabilities

Calc III



$f(x,y)$ = joint PDF of x & y.

$f(x_0,y_0)\,dA$ = probability that x & y are found inside the small tile dA near (x_0,y_0).

"volume of the column above dA".

$$\iint_R f(x,y)\,dA = \left(\begin{array}{c}\text{probability}\\ \text{that } x \text{ \& } y\\ \text{are inside } R\end{array}\right).$$

Ex. 2 The joint PDF for (X,Y) is:

$$f(x,y) = \begin{cases} Cxy, & (x,y) \text{ is in } R = [0,2]\times[0,3] \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the constant C that makes $f(x,y)$ a PDF.

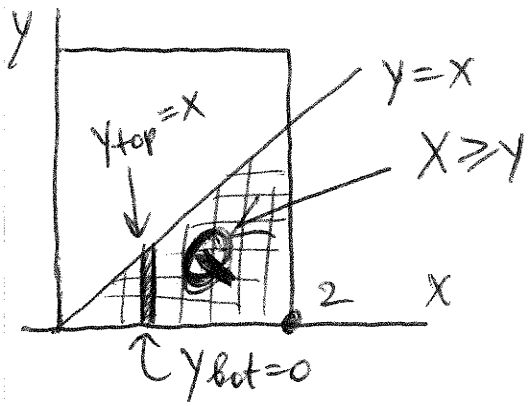
(b) Find $P(X \geq Y)$.

Sol'n: (a) $\left(\begin{array}{l}\text{Probability that}\\ (X,Y) \text{ is } \underline{\text{somewhere}}\\ \text{in } R\end{array}\right) = 1.$

$$\iint_R Cxy\,dA = 1 \quad \leftarrow \text{Find } C \text{ from this condition.}$$

$$1 = \int_0^3 \left(\int_0^2 Cx\,dx\right) y\,dy = C\int_0^2 x\,dx \int_0^3 y\,dy = C\cdot\frac{2^2}{2}\cdot\frac{3^2}{2}$$

$$\Rightarrow C = 1/9.$$



(b) $P(X \geq Y)$

$$= \iint_Q f(x,y)\,dx\,dy$$

$$= \int_0^2 \left(\int_0^x \frac{1}{9}xy\,dy\right)dx =$$

$$= \int_0^2 \left(\frac{x}{9} \cdot \frac{y^2}{2} \Big|_0^x \right) dx = \int_0^2 \frac{x}{9} \cdot \frac{x^2}{2} dx$$

$$= \frac{1}{18} \cdot \frac{x^4}{4} \Big|_0^2 = \boxed{\frac{2}{9}} \leftarrow \text{answer for (6).}$$

HW: 3, 9, 16; 27.

9 ← non/WA

3, 16: mass, CoM

27: probability