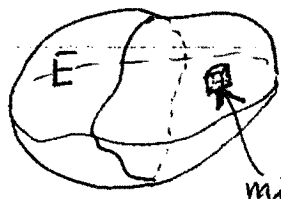


Sec. 15.6. Triple integrals

① Motivation and basic definition



Suppose we want to find the mass of a nonuniform solid E .

$$M = \sum_i m_i, \quad m_i = \rho(x_i, y_i, z_i) \Delta V$$

So

$$M = \lim_{\Delta V \rightarrow 0} \sum_i \rho(x_i, y_i, z_i) \Delta V$$

density ρ ΔV
volume of a small box around (x_i, y_i, z_i)

$$= \left[\iiint_E \rho(x, y, z) dx dy dz \right] \leftarrow \text{triple } \int \text{ over } E.$$

Other examples:

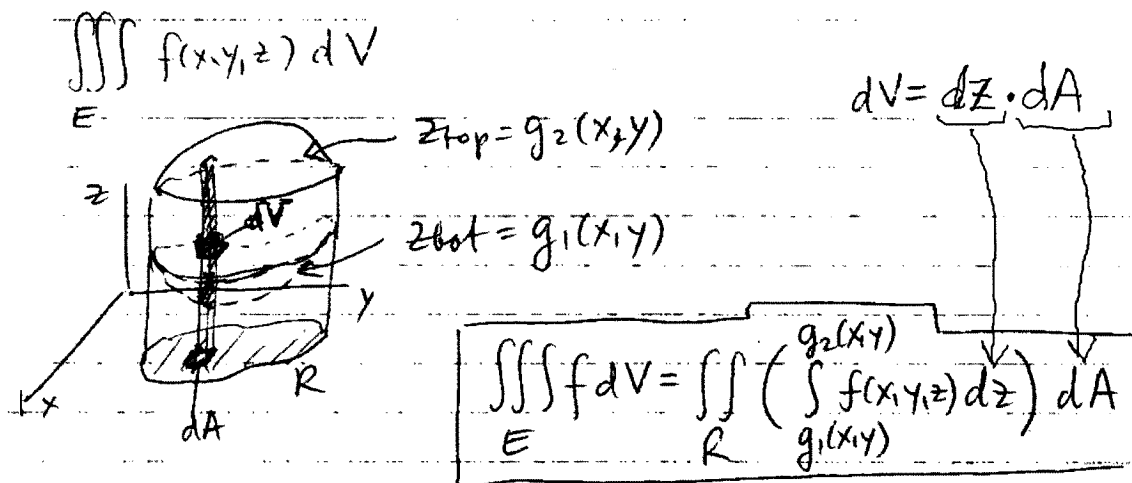
- Volume of $E = \sum_i \Delta V = \iiint_E 1 \cdot dV = \iiint_E dx dy dz$.
- Similar to Sec. 15.4, we can define x, y, z -coord's of the center of mass of E :

$$\bar{x}_c = \iiint_E x \rho(x, y, z) dV / M \quad (M = \iiint_E \rho(x, y, z) dV)$$
 and similar for $\bar{y}_c, \bar{z}_c$.
- If $p(x, y, z)$ is the joint probability density of quantities x, y, z , then

$$\iiint_E p(x, y, z) dV = \text{Probability that the simultaneous event } (x, y, z) \text{ is found within bounds of } E.$$
 (lower-case sigma)
 • If $\epsilon(x, y, z)$ is the density of electric charge of solid E , then $Q = \iiint_E \epsilon(x, y, z) dV = \text{total electric charge of } E$.

② Computing a $\iiint$ as an iterated integral.

Suppose solid E is bounded by surfaces $z_{\text{bot}} = g_1(x, y)$, $z_{\text{top}} = g_2(x, y)$ and its projection on xy -plane is a region R . Then:

$$\iiint_E f(x, y, z) dV$$


$$dV = dz \cdot dA$$

$$\boxed{\iiint_E f dV = \iint_R \left(\int_{g_1(x, y)}^{g_2(x, y)} f(x, y, z) dz \right) dA}$$

Thus, we reduced the problem of computing a triple int'l to two familiar steps:

- 1) Compute the single int'l over z ;
- 2) Compute the double int'l over R .

See in book Examples 2, 3, 5.

Often E can be viewed alternatively, e.g., as bounded by surfaces $y_{\text{bot}} = h_1(x, z)$, $y_{\text{top}} = h_2(x, z)$ and having a projection S on the xz -plane. (This is analogous to how one can view some regions in the xy -plane as Type I & Type II; Sec. 15.2.)

Then, if $f(x, y, z)$ is continuous over E , one has:

$$\text{in } xy\text{-plane} \rightarrow \iint_R \left(\int_{g_1(x, y)}^{g_2(x, y)} f(x, y, z) dz \right) dA = \underbrace{\quad}_{\text{in } (xy)}$$

$$\text{in } xz\text{-plane} \rightarrow \iint_S \left(\int_{h_1(x, z)}^{h_2(x, z)} f(x, y, z) dy \right) dA = \iint_T \left(\int_{\dots}^{\dots} f(x, y, z) dx \right) dA$$

$\leftarrow \text{in } (yz) \rightarrow$

Related Example in textbook is Ex. 4.

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→ Ex. 1 Describe the solid E defined by the integration limits. Then rewrite the given $\iiint$ as an iterated $\int$ in all other possible orders.

$$I = \int_{-2}^2 \int_{-1}^{3-y^2} \int_0^{1+z} f(x,y,z) dx dz dy$$

Note 1: Book's notation for solid $E = \{0 \leq x \leq 1+z, -1 \leq z \leq 3-y^2, -2 \leq y \leq 2\}$.

Note 2: # of permutations of dx, dy, dz is: $3 \cdot 2 \cdot 1 = 6$.

So we need to rewrite I in 5 ($= 6-1$) alternative orders.

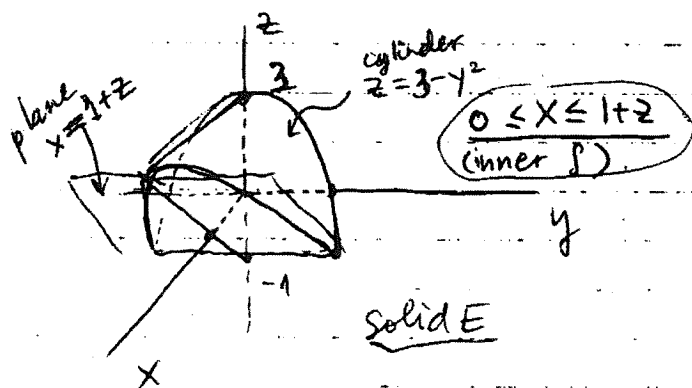
Sol'n: 1) Describe E . Sketch it first.

Recall that $\iint_R \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z) dz \right) dx dy$

defines a solid bounded by $z_{\text{bot}} = g_1(x,y)$, $z_{\text{top}} = g_2(x,y)$ over region R in xy -plane.

By analogy, solid E defined by the limits of I is:

- bounded by $x_{\text{bot}} = 0$ & $x_{\text{top}} = 1+z$ (inner limits);
- over some region T in yz -plane.



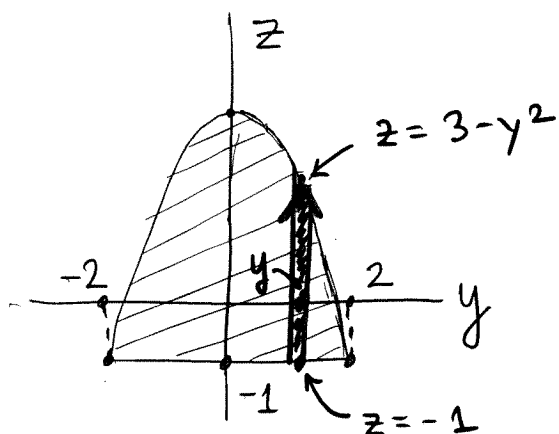
→ Thus:
When solving problems about triple integrals, one must SKETCH not only the 3D object, but also its projection on one of the coord. planes.

We will practice doing so in this Example.

! So, remember: You need 2 SKETCHes for SSS, not just 1.

We now sketch region T based on the limits of the (z, y) -integration and the 3D sketch.

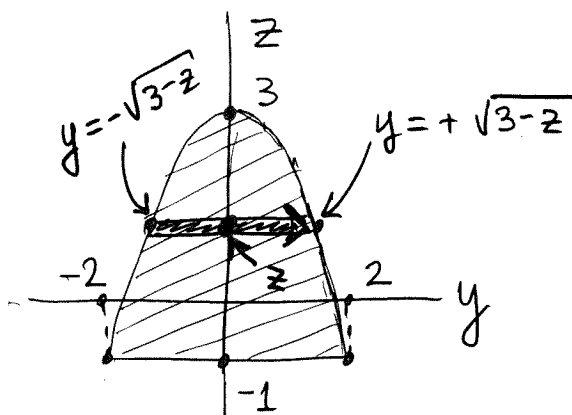
$$I = \int_{-2}^2 \int_{-1}^{3-y^2} \left(\int_0^{1+z} f \, dx \right) dz \, dy \Rightarrow \begin{aligned} -1 \leq z \leq 3-y^2 \\ -2 \leq y \leq 2 \end{aligned}$$



Region T set up as Type I.

We now need to set up the limits in the remaining 5 ways.

2) (a) Region T in the yz -plane as Type II



Region T as Type II.

Inner limits:

$$-\sqrt{3-z} \leq y \leq \sqrt{3-z}$$

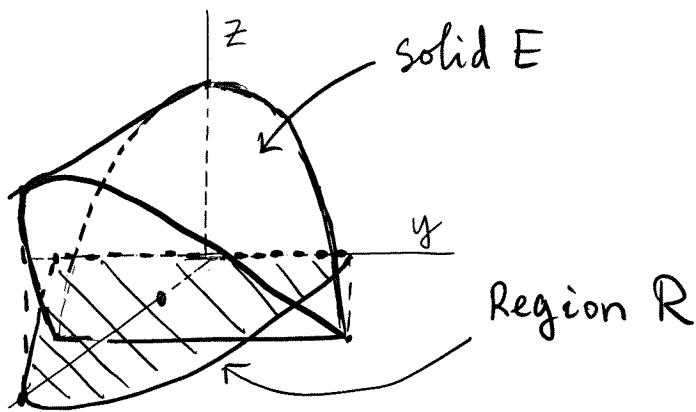
Outer limits:

$$-1 \leq z \leq 3$$

So $I =$

$$\int_{-1}^3 \int_{-\sqrt{3-z}}^{\sqrt{3-z}} \left(\int_0^{1+z} f \, dx \right) dy \, dz$$

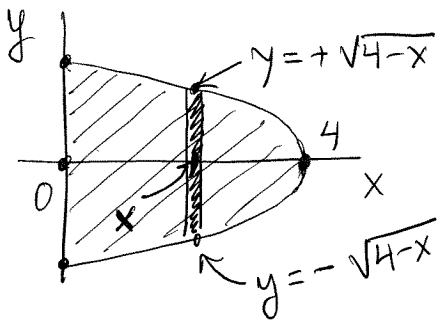
(b) Project solid E on region R in the xy -plane:
 We no longer can manipulate the original limits.
 We MUST USE THE 3D SKETCH to find new limits.



To find R , we need to find the projection of the intersection curve of $z = 3 - y^2$ & $x = 1 + z$ on the xy -plane. From the 3D sketch, we see that $z_{\text{top}} = 3 - y^2$, $z_{\text{bottom}} = x - 1$, so we are setting up $z_{\text{top}} = z_{\text{bot}}$ (as in Ex. 1 in Sec. 15.3) to find one of the boundaries of R :

$$3 - y^2 = x - 1 \Rightarrow x = 4 - y^2 \text{ (sideways parabola).}$$

The other boundary of R is $x = 0$ (see the 3D sketch).

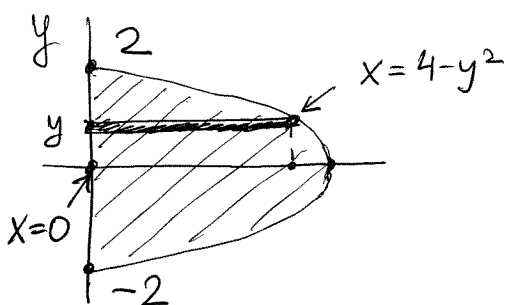


R as Type I

$$-\sqrt{4-x} \leq y \leq \sqrt{4-x}; \quad 0 \leq x \leq 4$$

$$I = \int_0^4 \int_{-\sqrt{4-x}}^{\sqrt{4-x}} \left(\int_{\substack{x-1 \\ z_{\text{bot}}}}^{\substack{3-y^2 \\ z_{\text{top}}}} f \, dz \right) dy \, dx$$

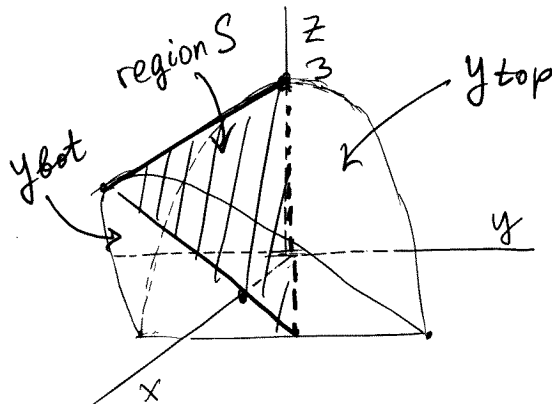
Now, R as Type II:



$$0 \leq x \leq 4 - y^2, \quad -2 \leq y \leq 2$$

$$I = \int_{-2}^2 \int_0^{4-y^2} \left(\int_{x-1}^{3-y^2} f \, dz \right) dx \, dy$$

(c) Project solid E on region S in xz -plane
 Again, we look at the 3D sketch to identify y_{top} & y_{bot} ; these come from cylinder $z = 3 - y^2$:



$$y_{top} = +\sqrt{3-z}$$

$$y_{bot} = -\sqrt{3-z}$$

Boundary of S comes from

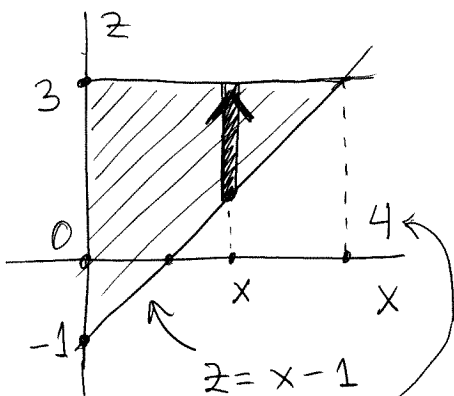
$$y_{top} = y_{bot}$$

$$+\sqrt{3-z} = -\sqrt{3-z} \Rightarrow$$

$$z = 3 \quad (\text{could have been found by looking at the sketch})$$

The other two boundaries of S come from the two planes

perpendicular to the xz -plane: $x=0$ & $z=x-1$.



intersection
 of $z=3$ &
 $z=x-1$

Region S as Type I:

$$x-1 \leq z \leq 3, \quad 0 \leq x \leq 4$$

So,

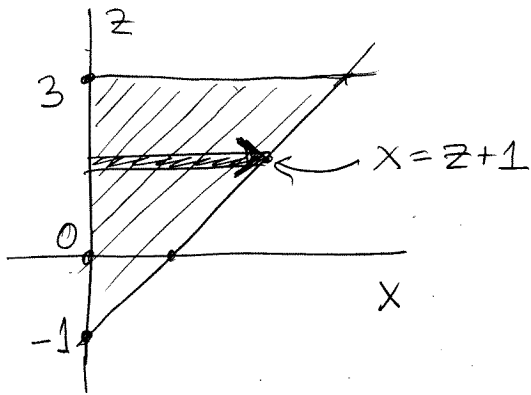
$$I =$$

$$\int_0^4 \int_{x-1}^3 \left(\int_{\underbrace{-\sqrt{3-z}}_{y_{bot}}}^{\underbrace{\sqrt{3-z}}_{y_{top}}} f \, dy \right) dz \, dx$$

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Region S as Type II:

$$0 \leq x \leq z+1, \quad -1 \leq z \leq 3$$



$$I =$$

$$\int_{-1}^3 \int_0^{z+1} \left(\int_{-\sqrt{3-z}}^{\sqrt{3-z}} f dy \right) dx dz$$

