

Sec. 15.9 Change of variables in multiple integrals

1.1 Motivation

Consider $\int \cos(x^2) 2x dx \equiv I$.

Usual way: $u = x^2$, $du = 2x dx$, etc.

Equivalent way: $x = \sqrt{u}$, $dx = \frac{dx}{du} \cdot du = \frac{1}{2\sqrt{u}} du$, $\Rightarrow$

$$I = \int \cos(u) \cdot \cancel{2\sqrt{u}} \cdot \frac{du}{\cancel{2\sqrt{u}}} = \int \cos u du.$$

In general:

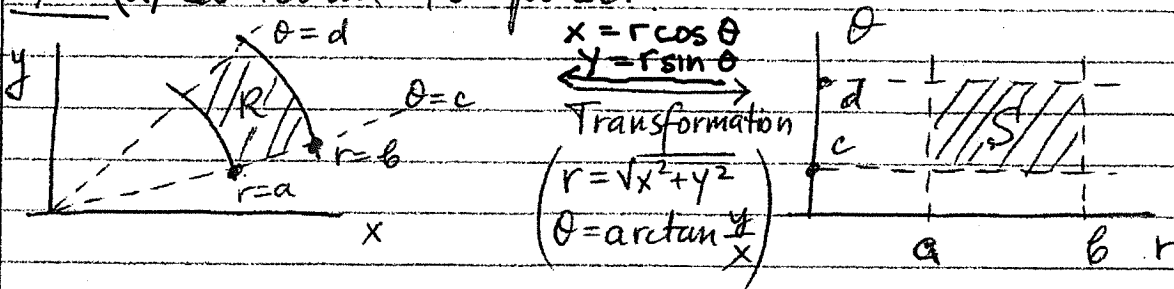
We seek a $x(u)$ such that this integral over u is simpler than the original integral over x .

$$\int f(x) dx = \int f(x(u)) \frac{dx}{du} \cdot du \quad (*)$$

Q: How does this generalize to double integrals?

1.2 Example of variable transformation

Ex. 1(a) Cartesian to polar



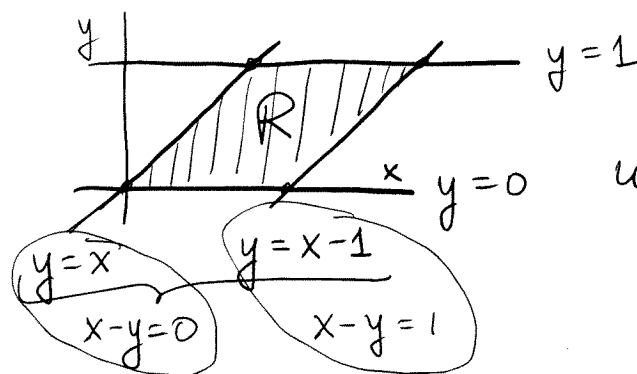
So this transformation takes a non-rectangular region R in xy -plane into a rectangle S in $r\theta$ -plane.

$$\iint_R f(x,y) (dx dy) = \iint_S f(x(r,\theta), y(r,\theta)) (r) dr d\theta \quad (**)$$

In this Section we pursue 2 Goals:

Goal ① Learn how to find a transformation $(x, y) \leftrightarrow (u, v)$ that transforms some region R in the (x, y) plane into a rectangle in the (u, v) -plane.

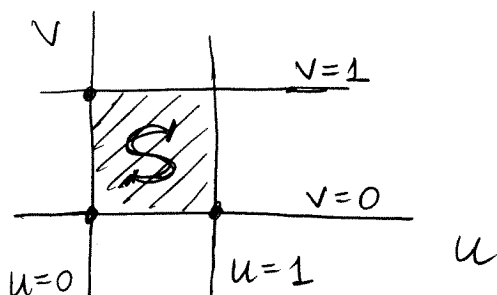
Example 1(b):



Bounds in (x, y) :

$$x-y=0, \quad x-y=1$$

$$y=0, \quad y=1$$



Bounds in (u, v) :

$$u=0, \quad u=1$$

$$v=0, \quad v=1$$

What we will do later is generalize Ex. 1(a) to the case where **R** is an arbitrary parallelogram.

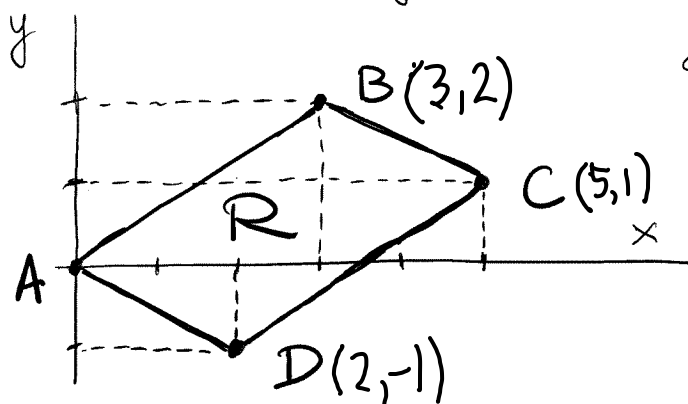
Goal ② Once we have found the desired transformation $(x, y) \leftrightarrow (u, v)$, we will learn how to "transform" $dx dy$, i.e.:

$$\iint_R f(x, y) (dx dy) = \iint_S \underbrace{f(x(u, v), y(u, v))}_{\text{Some } F(u, v)} \cdot \text{???} \cdot du dv$$

This "???" = r for $(x, y) \leftrightarrow (r, \theta)$, but how do we find it in general?

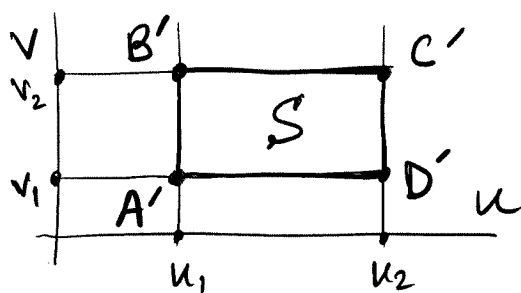
② Achieving Goal 1: Finding a transformation
from a parallelogram to a rectangle.

Ex. 2 Find a transformation $(x, y) \leftrightarrow (u, v)$
 such that a given parallelogram R in the xy-plane
 becomes a rectangle in the (u, v) -plane.



Sidenote: Parallelogram
 R is neither Type I
 nor Type II region,
 so integration over
 it would be complicated.

1) As Ex. 1a & 1b suggest, the required transfor-
 mation $(x, y) \leftrightarrow (u, v)$ is suggested by the
equations of the boundaries of R .



We seek:

$$\left. \begin{array}{l} u = \text{const along } AB \text{ \& } DC \\ v = \text{const along } AD \text{ \& } BC \end{array} \right\} \text{ of } R$$

(Just for convenience, we
 will do it in reverse order:
 first $v = \text{const}$, then $u = \text{const}$.)

$v = \text{const}$

along AD: $y = -\frac{1}{2}x$, or $x + 2y = 0$

along BC: $y = -\frac{1}{2}x + \text{const}$, or $x + 2y = \text{const}$

So $\boxed{v = x + 2y}$

↑
 will find
 later

$u = \text{const}$ along AB: $y = \frac{2}{3}x$, or $2x - 3y = 0$ along DC: $y = \frac{2}{3}x + \text{const}$, or $2x - 3y = \text{const}$ ↑
Will find
later

$$\boxed{u = 2x - 3y}$$

2) The above work has already given us two boundaries of S : on $A'D'$, $v = 0$ (see the work for " $v = \text{const}$ ")
on $A'B'$, $u = 0$ (see the work for " $u = \text{const}$ ")

It remains to find the "const"s for BC & DC.

To do so, pick any point on each of these sides.

On BC, pick $B = (3, 2)$; then $\underline{v = (x + 2y)}|_{@ (3, 2)} = 3 + 4 = \underline{7}$

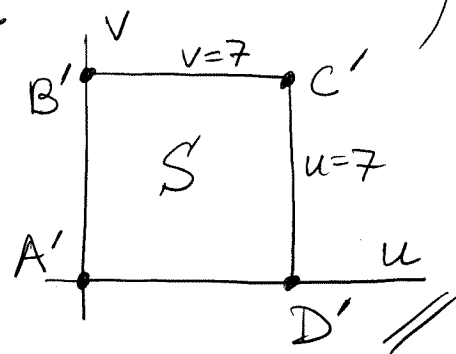
On DC, pick $D = (2, -1)$; then $\underline{u = (2x - 3y)}|_{@ (2, -1)} = 4 + 3 = \underline{7}$

(Note: The fact that these two constants are equal is a mere coincidence!)

Thus, the rectangle S

in (u, v) -plane is:

$$S = \{0 \leq u \leq 7, 0 \leq v \leq 7\}.$$

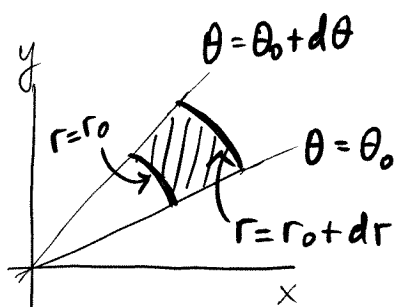


Note: We have found $u = x + 2y \equiv u(x, y)$
 $v = 2x - 3y \equiv v(x, y)$.

We will see later that we also need the inverse transformation $x = x(u, v)$, $y = y(u, v)$. It is found by solving for x, y : $x = (3u + 2v)/7$, $y = (2u - v)/7$.

③ Achieving Goal 2: Finding dA (= d"area")
under the transformation $(x,y) \leftrightarrow (u,v)$.

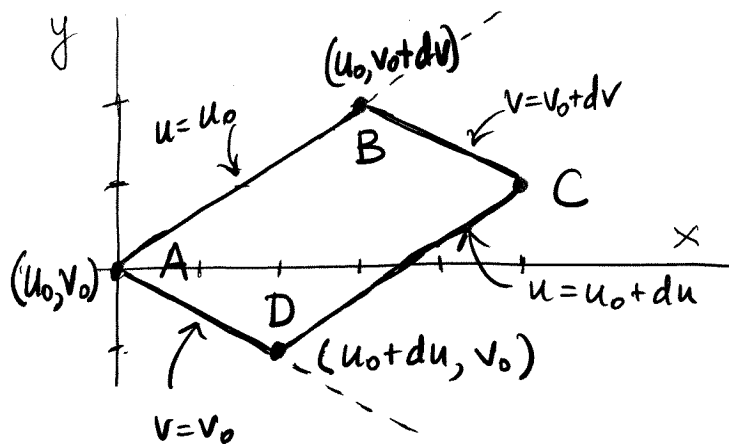
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We are again guided by Ex. 1,
the change Cartesian $\rightarrow$ Polar.
Namely, there we found dA
as the area of a small region

bounded by: $r=r_0, r=r_0+dr, \theta=\theta_0, \theta=\theta_0+d\theta$.

In complete analogy, we will find dA as the
area of a small parallelogram bounded by:



$$u=u_0, u=u_0+du$$

$$v=v_0, v=v_0+dv$$

1) From Sec. 12.4,

$$S_{ABCD} = |\vec{AD} \times \vec{AB}|$$

So need $\vec{AD}, \vec{AB}$ in terms of u, v .

$$\begin{aligned} 2) \vec{AD} &= \langle x_D, y_D \rangle - \langle x_A, y_A \rangle = \\ &\langle x(u_0+du, v_0), y(u_0+du, v_0) \rangle - \langle x(u_0, v_0), y(u_0, v_0) \rangle \end{aligned}$$

Let's focus on the x -coordinate first:

$$x(u_0+du, v_0+0) - x(u_0, v_0) \approx \frac{\partial x}{\partial u}(u_0, v_0) \cdot du + \frac{\partial x}{\partial v}(u_0, v_0) \cdot 0$$

And similarly for y . Then:

$$\boxed{\vec{AD} \approx \left\langle \frac{\partial x}{\partial u} \cdot du, \frac{\partial y}{\partial u} \cdot du \right\rangle}$$

Similarly,

$$\begin{aligned}\vec{AB} &= \langle x_B, y_B \rangle - \langle x_A, y_A \rangle \\ &= \langle x(u_0, v_0 + dv), y(u_0, v_0 + dv) \rangle - \langle x(u_0, v_0), y(u_0, v_0) \rangle \\ &\approx \left\langle \frac{\partial x}{\partial v} \cdot dv, \frac{\partial y}{\partial v} \cdot dv \right\rangle\end{aligned}$$

3) Put this into the formula:

$$\begin{aligned}S_{ABCD} &= |\vec{AD} \times \vec{AB}| = \left| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u du & y_u du & 0 \\ x_v dv & y_v dv & 0 \end{vmatrix} \right| \\ &= \left| \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \cdot du dv \right| = \left| \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \right| \cdot du dv\end{aligned}$$

Annotations: "length" points to the magnitude bars; "det" points to the determinant of the Jacobian matrix; "abs. value" points to the absolute value bars.

Thus:

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$$(dx dy) \leftrightarrow \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \cdot du dv \quad (J)$$

Change of dA under $(x, y) \leftrightarrow (u, v)$.

Note 1: Need the abs. value, $|\dots|$, since the area cannot be negative.

Note 2: The quantity J is called **the Jacobian**

$$J = \left| \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \right| \equiv \frac{\partial(x, y)}{\partial(u, v)}$$

after the German mathematician Carl Jacobi.

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Note 3 Rows & columns of J can be switched (this convention is used in the textbook).

Note 4 Recall that in this course, we denote

$\vec{r} \equiv \langle x, y \rangle$. Since in this case, $x = x(u, v)$ and $y = y(u, v)$, we have

$$\vec{r} = \langle x(u, v), y(u, v) \rangle \equiv \vec{r}(u, v).$$

Then in our derivation,

$$\vec{AD} = \langle x_u du, y_u du \rangle = \langle x_u, y_u \rangle du = \vec{r}_u \cdot du$$

$$\vec{AB} = (\text{similarly}) \vec{r}_v \cdot dv.$$

I.e., the side where only u changes $= \vec{r}_u \cdot du$

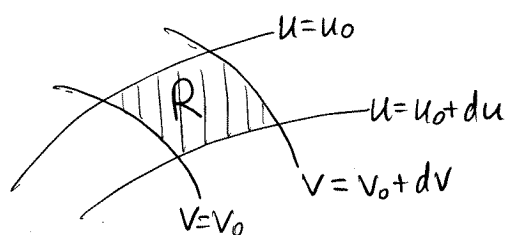
the side where only v changes $= \vec{r}_v \cdot dv$

(we will use this later & in Sec. 16.6)

$$\boxed{S_{ABCD} = |\vec{AD} \times \vec{AB}| = |\vec{r}_u \cdot du \times \vec{r}_v \cdot dv|}$$
$$\boxed{= |\vec{r}_u \times \vec{r}_v| \cdot du dv} \leftarrow \text{Will use this in 16.6 \& 16.7.}$$

abs. value

Note 5 If the transformation $(x, y) \leftrightarrow (u, v)$ is such that a region R with curved boundaries (such as the sector in polar



coord's; Ex. 1) is transformed into a

rectangle S' , our derivation does

not change, because a very small

curved region R is approximately

a parallelogram!

Note 6 In the familiar example of changing from Cartesian to polar coordinates,

$$x = \overset{u}{r} \overset{v}{\cos \theta}, \quad y = r \sin \theta, \quad \text{one has:}$$

$$J = \left| \begin{vmatrix} x_r & y_r \\ x_\theta & y_\theta \end{vmatrix} \right| = \left| \begin{vmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{vmatrix} \right| =$$

$$\left| \cos \theta \cdot r \cos \theta - \sin \theta \cdot (-r \sin \theta) \right| = \left| r \cos^2 \theta + r \sin^2 \theta \right| = r,$$

the familiar result.

④ A slight variation: transforming into a circle

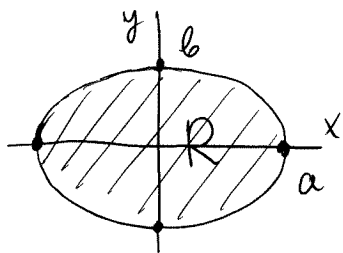
We chose to transform a given region R into a rectangle because it is very easy to integrate over a rectangle.

But by now we are also comfortable integrating over a circle (where we can use polar coordinates).

So sometimes it is easier to transform a given region into a circle instead of a rectangle.

Ex. 3(a)

Use transformation $x = a \cdot u$
 $y = b \cdot v$



to express

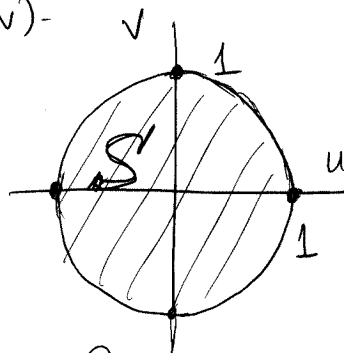
$\iint_R f(x,y) dx dy$ in the
 (u,v) -coordinates.

Sol'n: 1) Find what R transforms to:

$$R: \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right) \text{ (Cartesian eq. of ellipse, Sec. 12.6)}$$

$$\Rightarrow S: \frac{(a \cdot u)^2}{a^2} + \frac{(b \cdot v)^2}{b^2} = 1 \Rightarrow \frac{a^2 u^2}{a^2} + \frac{b^2 v^2}{b^2} = 1$$

$$\Rightarrow \underline{u^2 + v^2 = 1} \leftarrow \text{circle in } (u,v)\text{-plane}$$



2) Change dA :

$$dx dy = J \cdot du dv$$

↑
Jacobian

$$J = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = \begin{vmatrix} \frac{\partial(au)}{\partial u} & \frac{\partial(bv)}{\partial u} \\ \frac{\partial(au)}{\partial v} & \frac{\partial(bv)}{\partial v} \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$$

$$= a \cdot b, \Rightarrow$$

$$dx dy = ab \cdot du dv$$

$$3) \iint_R f(x,y) dx dy = \iint_S f(au, bv) \cdot (ab) \cdot du dv.$$

One can now switch $(u,v) \rightarrow (u = r \cos \theta, v = r \sin \theta)$ if need be.

Ex. 3(b) Find the area of the ellipse R from part (a).

Sec. 15.2 (end)

Sol'n: $A_R = \iint_R 1 \cdot dx dy \xrightarrow{(a)}$

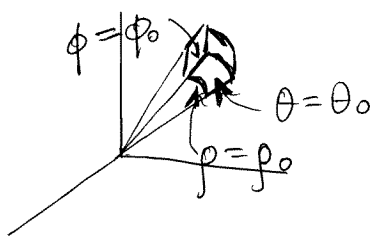
$$\begin{aligned}
 &= \iint_S 1 \cdot (ab) \, du \, dv = ab \cdot \underbrace{\iint_S 1 \cdot du \, dv}_{\text{area of circle } S} \\
 &= ab \cdot \pi \cdot 1^2.
 \end{aligned}$$

Thus, $\boxed{A_R = \pi \cdot ab}$.

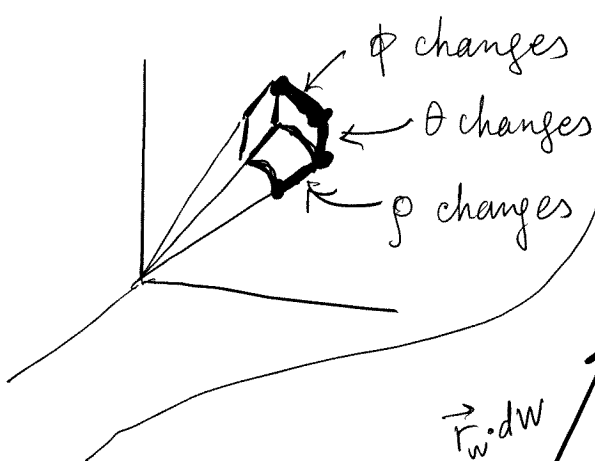
⑤ Change of variables in 3D

Question: How does $(dx dy dz)$ transform under $(x, y, z) \leftrightarrow (u, v, w)$?

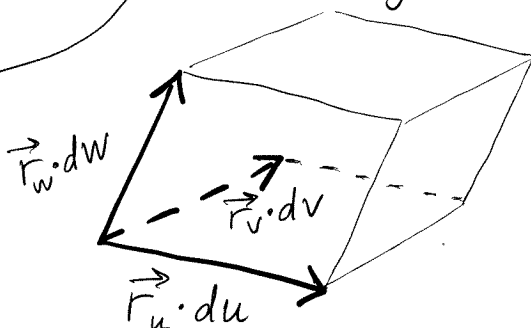
We will proceed by analogy with the change Cartesian $\rightarrow$ spherical, $(x, y, z) \leftrightarrow (\rho, \phi, \theta)$:



We found dV as the volume of a small box, where on edge of the box, only one coordinate would change (ρ , or ϕ , or θ):



Similarly, in (u, v, w) , we consider a box where along each edge, only 1 coordinate changes:



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This is similar to Note 4 on p. 26-7:

"the side where only u changes" $= \vec{r}_u \cdot d\vec{u}$, etc.

Then $dV =$ the volume of this box (Sec. 12.4, triple product)

$$= \left| (\vec{r}_u \cdot d\vec{u}) \bullet (\vec{r}_v dv \times \vec{r}_w dw) \right| = \left| \vec{r}_u \bullet (\vec{r}_v \times \vec{r}_w) \right| du dv dw$$

abs. value

$$\text{Thus, } J = \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix} = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right|$$

abs. value $\rightarrow$ $\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right|$ $\leftarrow$ abs. value

$\nwarrow$ det

$$dV = dx dy dz = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \cdot du dv dw$$

For example, when $(u, v, w) \equiv (\rho, \phi, \theta)$, $x = \rho \sin \phi \cos \theta$
then $y = \rho \sin \phi \sin \theta$
 $z = \rho \cos \phi$

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} x_\rho & y_\rho & z_\rho \\ x_\phi & y_\phi & z_\phi \\ x_\theta & y_\theta & z_\theta \end{vmatrix} = \rho^2 \sin \phi$$

$\nwarrow$ Ex. 4 in book

$$\Rightarrow dx dy dz = \left| \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} \right| d\rho d\phi d\theta = \rho^2 \sin \phi d\rho d\phi d\theta.$$

