Sec. 16.5. Curl and divergence.

① Definitions.

Recall the gradient: $\vec{\nabla} f(x,y,z) = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$.

Define vector $\boxed{\vec{\nabla} \equiv \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle}$.

Name: $\vec{\nabla}$ = nabla (not Greek)

It may look formal but:

- Has all properties of a vector hence can be treated as such (power of abstraction!);
- In higher-level (grad) courses it is shown that in some way, $\vec{\nabla}$ behaves just like a regular vector $\langle a, b, c \rangle$.

$\vec{\nabla} f$ shows how vector $\vec{\nabla}$ acts on a scalar function f.

How does $\vec{\nabla}$ act on a vector function $\vec{F} \equiv \langle P, Q, R \rangle$?

Curl:

$$\vec{\nabla} \times \underbrace{\vec{F}}_{\langle P, Q, R\rangle} = \text{curl}\, \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

(see Ex. 1 in book for numbers).

Divergence:

$$\vec{\nabla} \cdot \vec{F} = \text{div}\, \vec{F} = \frac{\partial}{\partial x} P + \frac{\partial}{\partial y} R + \frac{\partial}{\partial z} R.$$

(see Ex. 4 in book).

② Meaning of curl & div.

Ex. 1 Find curl & div of $\vec{F} = \vec{r} \equiv \langle x, y, z \rangle$

0) Draw:

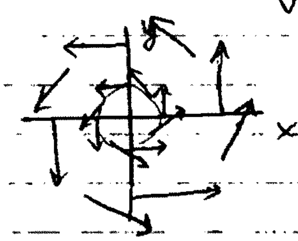
This vector field diverges from (0,0,0). (Similarly to the 2D Ex. 1/Notes for Sec. 16.1.)

1) $\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{i}\,(\cancel{z_y}^{0} - \cancel{y_z}^{0}) - \vec{j}\,(\cancel{z_x}^{0} - \cancel{x_z}^{0})$
$+ \vec{k}\,(\cancel{y_x - x_y}^{0}) = \vec{0}\,!$

2) $\operatorname{div}\vec{F} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 3.$

Ex. 2 Find curl & div of $\vec{F} = \langle -y, x, 0 \rangle$.

0) Sketch:
(Ex. 2 in Sec. 16.1)
(Notes)



This field swirls around origin.

1) $\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = \vec{i}\,(0 - \cancel{x_z}^{0}) - \vec{j}\,(0 + \cancel{y_z}^{0})$
$+ \vec{k}\,(x_x + y_y) = 2\vec{k}.$

$\operatorname{curl}\vec{F} = 2\vec{k}$ — axis of rotation ← The curl is directed along the axis of rotation and its sign follows the right-hand rule (see Sec. 12.4.).

2) $\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}0 = 0$

Moral:
- div $\vec{F}$ measures the tendency of $\vec{F}$ to diverge from its source (or to converge into a sink).
  E.g., an expanding gaseous fireball diverges from its center.
- The ability to diverge (converge) is related to the ability to expand (or to contract). Therefore, if with div $\vec{F} = 0$, $\vec{F}$ describes an incompressible motion (e.g, of a fluid).

- A purely diverging $\vec{F}$ is <u>irrotational</u>, i.e. $\text{curl}\,\vec{F} = \vec{0}$. (Ex. 1).
- <u>curl $\vec{F}$</u> measures the <u>tendency of $\vec{F}$ to rotate</u>. (Irrotational field has $\text{curl}\,\vec{F} = \vec{0}$)
- A purely rotational field has zero divergence ($\text{div}\,\vec{F} = 0$); Ex. 2.

③ <u>Compositions of two $\vec{\nabla}$'s.</u>

(a) Let $\vec{F} = \text{curl}\,\vec{G} \equiv \vec{\nabla} \times \vec{G}$ for some $\vec{G}$.
Then $\text{div}\,\vec{F} = \vec{\nabla} \cdot \underbrace{(\vec{\nabla} \times \vec{G})}_{\text{Sec12.4} \to \text{"1"}\vec{\nabla}} = 0$ ← Sec. 12.3

(verified by direct calculation in the book)

<u>Conclusion:</u> A given $\vec{F}$ can be represented as $\vec{F} = \text{curl}\,\vec{G}$ for some $\vec{G}$ <u>if and only if</u> $\text{div}\,\vec{F} = 0$.
The proof is above.
See Ex. 5 in book for ##.

(b) Let $\vec{F} = \vec{\nabla} f$ for some f.
Then
$$\text{div}\,\vec{F} = \vec{\nabla} \cdot \vec{\nabla} f = \frac{\partial}{\partial x} f_x + \frac{\partial}{\partial y} f_y + \frac{\partial}{\partial z} f_z$$
$$\equiv f_{xx} + f_{yy} + f_{zz} \equiv \nabla^2 f$$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

is called <u>Laplacian</u> and plays a major role in <u>all</u> branches of physics and engineering.

(c) Again, let $\vec{F} = \vec{\nabla} f$. Sec. 16.1: $\vec{F}$ is a conservative field

Then

$$\text{curl } \vec{F} = \vec{\nabla} \times \underbrace{\vec{\nabla} f}_{\text{"}\parallel\text{"} \vec{\nabla}} = \vec{0}$$ ← Sec. 12.4

(verified by direct calculation in the book)

Conclusion: $\boxed{\text{A given } \vec{F} \text{ can be represented as } \vec{F} = \vec{\nabla} f \text{ for some } f \text{ if and only if } \text{curl } \vec{F} = \vec{0}.}$!!!

See Ex. 2, 3(a) in book.

Will study this again in Sec. 16.3 & 16.4.

④ Elementary properties

$$\text{div}\,(\vec{F} + \vec{G}) = \text{div}\,\vec{F} + \text{div}\,\vec{G}$$
$$\text{curl}\,(\vec{F} + \vec{G}) = \text{curl}\,\vec{F} + \text{curl}\,\vec{G}.$$

Other properties: see problems around ## 25-30 after Sec. 16.5; they are proved using the product rule.

⑤ Note about some incorrect notations.

Recall that $\text{div}\,\vec{F} = \vec{\nabla} \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle P, Q, R \rangle$ $= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$. Note the order of the derivative and the function in each term: e.g., in $\frac{\partial}{\partial x} P$: the derivative is written before the function! (This is how we have always written derivatives in this course when we used the $\frac{\partial}{\partial x}$ etc. notations.) However, for an unknown reason, I have seen many students write:

BAD! → $\text{div}\,\vec{F} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$ ← THIS IS WRONG!

The notation $\frac{\partial P}{\partial x}$ has a clear meaning: $\frac{\partial}{\partial x}$ is taken of P. On the other hand, the notation $P\frac{\partial}{\partial x}$ means something totally different, that we did not study in this course!

So, do not use this wrong notation in either $\text{div}\,\vec{F}$ or $\text{curl}\,\vec{F}$!