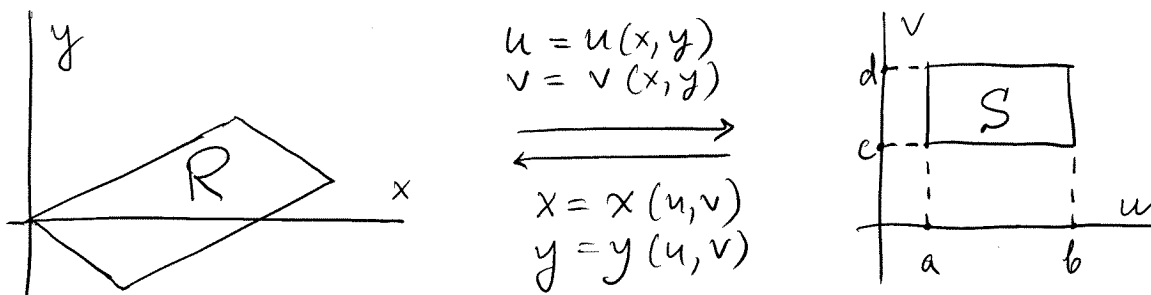


Sec. 16.6: Parametric surfaces

27-1

① Motivation and definition

We have seen that sometimes, a change of variables

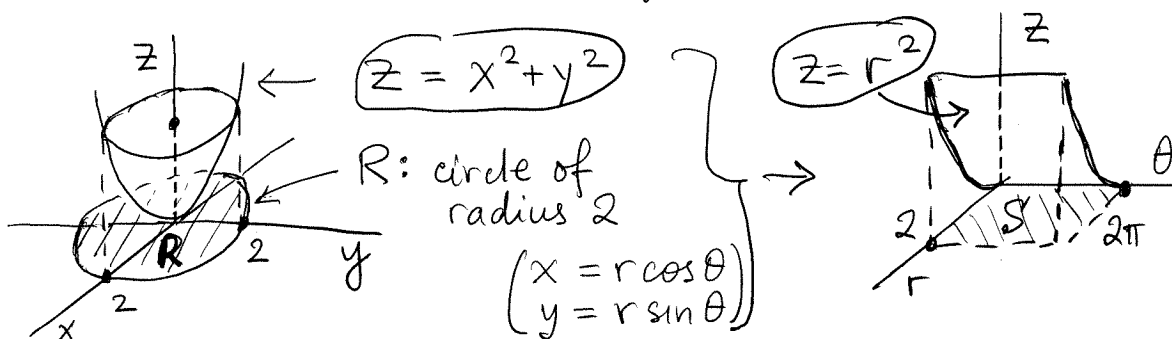


transforms a "complicated" region R in the xy -plane into a simpler region S in the uv -plane.

Then, when we consider a **surface** $z = z(x, y)$, it may be convenient to also define z in terms of the "convenient" coordinates (u, v) :

$$z = z(x(u, v), y(u, v)) \equiv f(u, v).$$

This will change the surface from being defined over a "complicated" region R into a (differently shaped) surface over the simpler region S :



So from now on, we will consider parametric surfaces defined as:

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$

Note 1: This is conceptually analogous to how we defined parametric curves ($x = x(t), y = y(t), z = z(t)$) in Chap. 13. On a curve, one has 1 Degree of Freedom: t . On a surface, one has 2 Degrees of Freedom: u & v .

Note 2: Recall that in this course we always use the notation $\vec{r} \equiv \langle x, y, z \rangle$. Since x, y, z all depend on (u, v) , we can now say that a parametric surface is defined by the vector equation $\vec{r} = \vec{r}(u, v)$, which is an equivalent form of $\langle x, y, z \rangle = \langle x(u, v), y(u, v), z(u, v) \rangle$.

In the Example above, the paraboloid $z = x^2 + y^2$, or $\vec{r} = \langle x, y, x^2 + y^2 \rangle$, can be written in parametric form as:

$$\vec{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, r^2 \rangle.$$

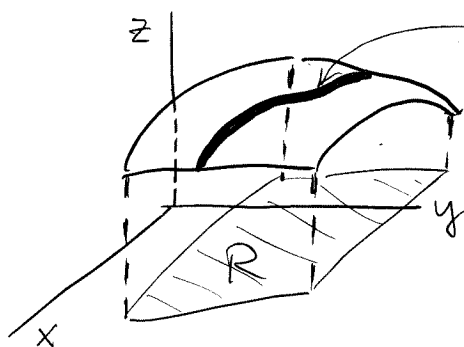
② Tangent plane, normal vector, and surface area of a parametric surface

a Setup: Let only one variable change in the parametric surface $\vec{r} = \langle x(u, v), y(u, v), z(u, v) \rangle$.

Say, $v = v_0$, u varies; then

$$\vec{r} = \langle x(u, v_0), y(u, v_0), z(u, v_0) \rangle$$

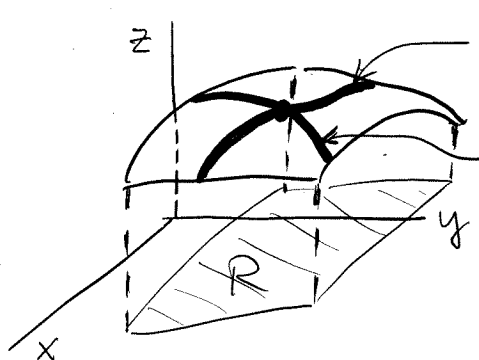
For a given v_0 , this is exactly the equation of a parametric curve (with $t \rightarrow u$) from Chap. 13.



$$\vec{r} = \langle x(u, v_0), y(u, v_0), z(u, v_0) \rangle \\ \equiv \vec{r}(u, v_0)$$

This curve lies on the parametric surface

$$\vec{r} = \vec{r}(u, v).$$

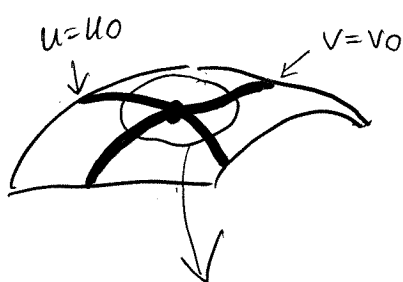


$$\vec{r} = \vec{r}(u, v_0) \\ \vec{r} = \vec{r}(u_0, v)$$

Similarly, for any fixed $u = u_0$, the

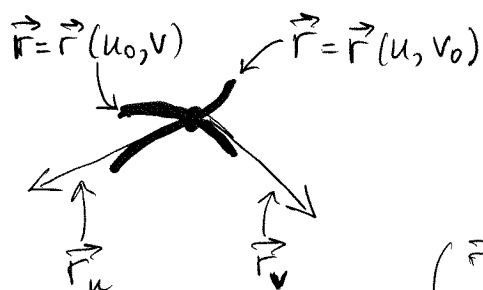
curve $\vec{r} = \vec{r}(u_0, v)$ also lies on this surface.

[b] Tangent plane & normal vector



Let us zoom in on the intersection of the curves

$$\vec{r} = \vec{r}(u, v_0) \text{ \& \> } \vec{r} = \vec{r}(u_0, v) \\ \text{near point } (u_0, v_0).$$



By Sec. 13.2, the tangent vectors to these curves are: $\vec{T}_u$ & $\vec{T}_v$.

$$\left(\vec{T}_u \equiv \frac{d\vec{r}(u, v_0)}{du}, \vec{T}_v \equiv \frac{d\vec{r}(u_0, v)}{dv} \right)$$

Since :

- $\vec{r}(u, v_0)$ & $\vec{r}(u_0, v)$ lie on surface $\vec{r}(u, v)$;
 - $\vec{r}_u$ is tangent to $\vec{r}(u, v_0)$; $\vec{r}_v$ is tangent to $\vec{r}(u_0, v)$;
- $\Rightarrow$ $\vec{r}_u$ & $\vec{r}_v$ lie in the plane tangent to $\vec{r}(u, v)$
(at point (u_0, v_0)).

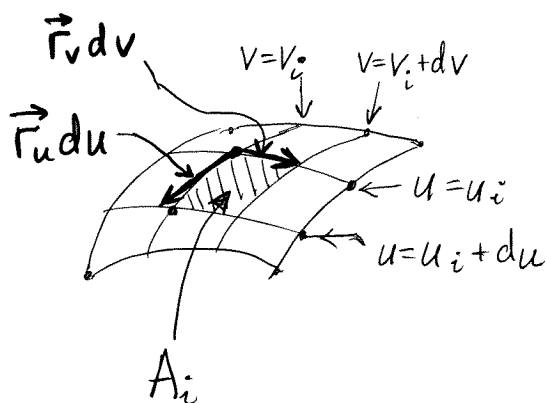
Then, by Sec. 12.4,

$$\boxed{\vec{n} = \vec{r}_u \times \vec{r}_v}$$

is the normal vector to surface $\vec{r} = \vec{r}(u, v)$.

Note that the tangent plane at (u_0, v_0) can be found from its two ingredients: the point $\vec{r}(u_0, v_0)$ and the normal vector $\vec{n}$ (see Ex. 9 in the book).

[C] Surface area



Suppose we want to find the area of a patch of the surface $\vec{r} = \vec{r}(u, v)$.

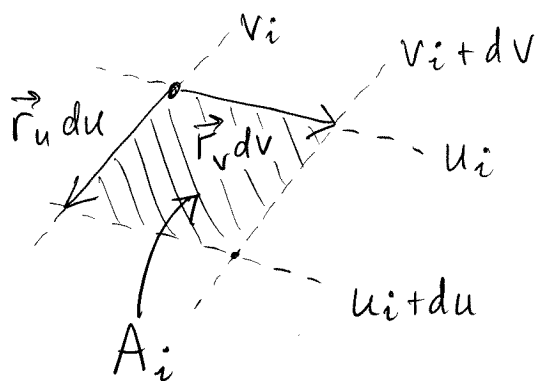
We can draw on it a "curved mesh" by lines $u = u_i (= \text{const})$, $v = v_i (= \text{const})$.

This mesh subdivides the surface into small tiles with areas A_i . Then

$$\text{Total area} = \sum_i A_i$$

To find A_i , we recall from Note 4 on p. 26-7

(see also the figure at bottom of p. 26-10)



the tile is a parallelogram with sides $\vec{r}_u du$, $\vec{r}_v dv$. In the same Note 4, we found:

$$A_i = \left| \vec{r}_u du \times \vec{r}_v dv \right| \Big|_{@ (u_i, v_i)}$$

$$\Rightarrow \boxed{A_i = \left| \vec{r}_u \times \vec{r}_v \right| \cdot du dv \Big|_{@ (u_i, v_i)}}$$

Then the total area of the patch of the surface $\vec{r} = \vec{r}(u, v)$ "over" the region S in the uv -plane is:

$$A_{\text{total}} = \sum_i A_i \longrightarrow \iint_S \left| \vec{r}_u \times \vec{r}_v \right| du dv.$$

See Ex. 10 in the book and also Ex. 3(b) below.

(See also Fig. 14 in the book for a better visualization of a curved mesh subdividing a surface.)

Note: For Cartesian surfaces $z = f(x, y)$, one can take $(u, v) \equiv (x, y)$. Then

$$\vec{r}_u \equiv \vec{r}_x = \langle x, y, f(x, y) \rangle_x = \langle 1, 0, f_x(x, y) \rangle$$

$$\vec{r}_v \equiv \vec{r}_y = \langle x, y, f(x, y) \rangle_y = \langle 0, 1, f_y(x, y) \rangle,$$

and (see the last topic of Sec. 16.6 in the book):

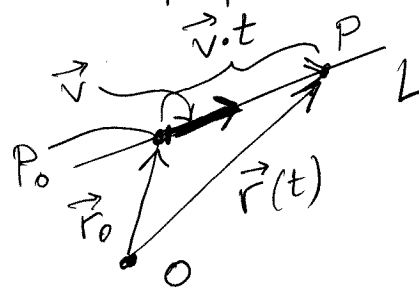
$$A_{\text{total}} = \iint_R \sqrt{1 + f_x^2 + f_y^2} dx dy.$$

MUST see Ex. 11 in the book.

③ Parametric eqs. of a plane

Recall from Sec. 12.5-A how the eq. of a line was derived:

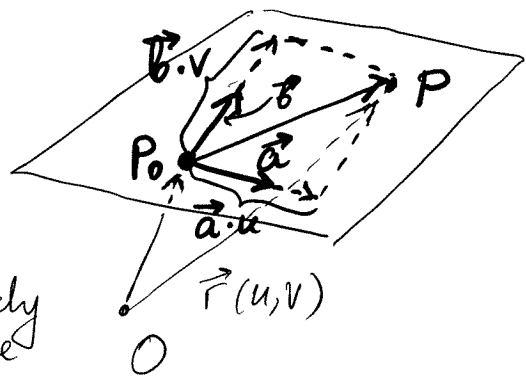
$$\vec{OP} = \vec{OP}_0 + \underbrace{\vec{P}_0P}_{\substack{\text{going along} \\ L}} \quad \substack{\uparrow \\ \text{some pt.} \\ \text{on } L}$$



$\vec{r} = \vec{r}_0 + \vec{v} \cdot t$; parameter t describes the 1 Degree of Freedom along the line.

Similarly, when $\vec{P}$ is on a plane "made" by vectors $\vec{a}$ and $\vec{b}$,

$$\vec{OP} = \vec{OP}_0 + \underbrace{\vec{P}_0P}_{\substack{\text{staying entirely} \\ \text{on the plane}}} \quad \substack{\uparrow \\ \text{some} \\ \text{pt. on plane}}$$



In the Addendum to Sec. 12.4 (posted online), we showed that if some two non-parallel vectors $\vec{a}$ & $\vec{b}$ "make" a plane, then (any) vector $\vec{P}_0P$ in that plane can be written as:

$$\vec{P}_0P = \vec{a} \cdot u + \vec{b} \cdot v \quad (\text{see the figure above})$$

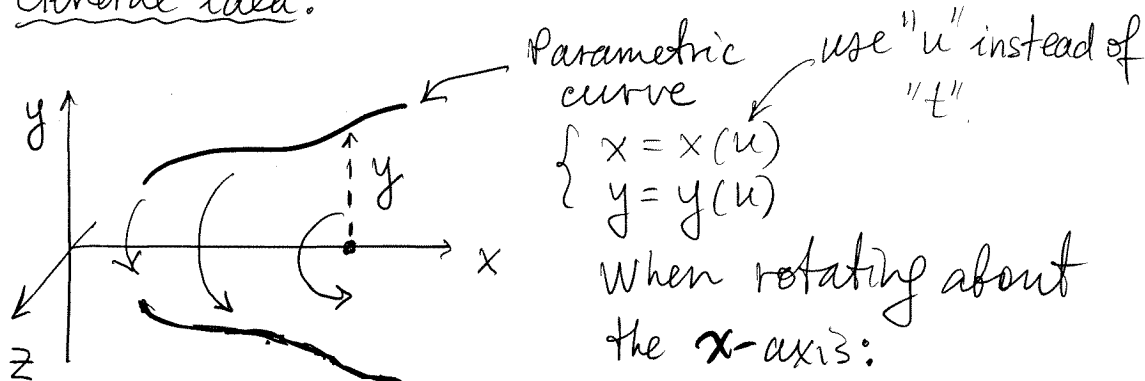
for some numbers u, v . Then:

$\vec{r}(u, v) = \vec{r}_0 + \vec{a} \cdot u + \vec{b} \cdot v$ is the param. eq. of a plane. Parameters u & v describe the 2 Degrees of Freedom of motion on a plane.

④ Parametric eqs. of Surfaces of Revolution

27-7

- General idea:



Thus, the parametric eqs. of the Surface of Revolution obtained when a curve is rotated about the

x -axis are:

$$\begin{cases} x = x(u) \\ y = y(u) \cdot \cos v \\ z = y(u) \cdot \sin v \end{cases}$$

$x = x(u)$ ← stays the same

$y = y(u) \cdot \cos v$ ← rotation: parametric eqs. of a circle

$z = y(u) \cdot \sin v$ ← radius of rotation

And similarly for surfaces obtained by rotation of curves about the y - and z -axes.

Note: Do NOT follow the "method" on Surfaces of Revolution presented in the book. You will be unable to obtain a new result of Ex. 4 below with the book's "method". You will also be unable to do Lab 7. Thus, using that "method" on a test will earn only 1 point for the effort.



Ex. 1 Find the parametric eqs. of cylinder
 $x^2 + y^2 = 4$.



Sol'n: 1) This cylinder is obtained by rotating the line $x=2$ in the xz -plane about the z -axis.

2) Need parametric eqs. of this line:

$$\begin{cases} x = 2 \\ z = u \leftarrow \text{any number} \end{cases}$$

3) Rotate about z -axis

$$\begin{cases} x = 2 \cdot \cos v \\ y = 2 \cdot \sin v \\ z = u \leftarrow \text{stays the same} \end{cases}$$

Parametric eqs. of cylinder w/ z -axis.

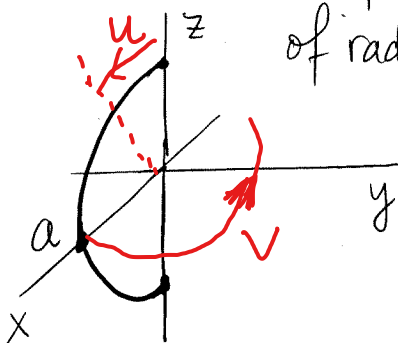
Note: • Rename: $v = \theta$, $u = z$;

• Allow radius "2" to be any number, "r".

Then these eqs become: $\begin{cases} x = "r" \cdot \cos \theta \\ y = "r" \sin \theta \\ z = "z" \end{cases}$

These are cylindrical coordinates (not surprisingly).

Ex. 2 Find parametric eqs. of a sphere of radius a and centered at the origin.



Sol'n: 1) Such a sphere is obtained by rotating a semi-circle (shown) about the z -axis.

(Note that if we rotate a full circle by 2π about the z -axis, we will "cover" the sphere twice.)

2) Parametric eqs. of the semi-circle:

$$\begin{cases} x = a \cdot \sin u \\ z = a \cdot \cos u, \quad 0 \leq u \leq \pi \end{cases} \quad \leftarrow \text{half-circle}$$

(We chose the $\sin$ & $\cos$ in this order for the convenience of a later step.)

3) Rotate about the z -axis:

$$\begin{cases} x = (a \cdot \sin u) \cdot \cos v \\ y = (a \cdot \sin u) \cdot \sin v \\ z = a \cdot \cos u \end{cases} \quad \leftarrow \begin{array}{l} \text{Parametric} \\ \text{eqs. of a} \\ \text{sphere of radius } a. \end{array}$$

$\leftarrow$ stays the same

Rename: $a = \rho$, $u = \phi$, $v = \theta$;

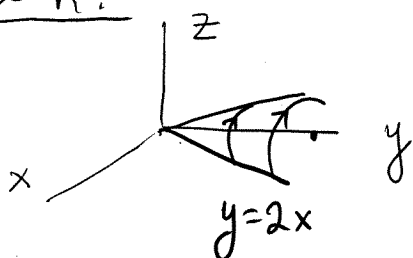
then we recover spherical coord's (not surprisingly).

Incidentally, the Note at the top of this page explains why the range for ϕ is: $0 \leq \phi \leq \pi$ (Sec. 15.8).

Ex. 3(a) Find param. eqs. of a half-cone

$$y = 2 \sqrt{x^2 + z^2}, \quad (y \geq 0)$$

Sol'n:



1) We first recognize that this (half-)cone has the y -axis as its axis.

Thus, it can be obtained by rotating the ray: $y = 2x, x \geq 0$, about the y -axis.

2) Parametric eqs. of the ray:

$$\begin{cases} x = u \\ y = 2u, u \geq 0. \end{cases}$$

(Recall - from HW for Sec. 13.1 - that any Cartesian eq. $y = f(x)$ can be written parametrically as: $x = t, y = f(t)$.)

3) Rotate about the y -axis:

$$\begin{cases} x = u \cdot \cos v \\ y = 2u \leftarrow \text{stays the same} \\ z = u \cdot \sin v \end{cases}$$

Parametric eqs. of a cone around the y -axis.

Equivalently, $\vec{r}_{\text{cone}}(u, v) = \langle u \cdot \cos v, 2u, u \cdot \sin v \rangle$.

Ex. 3(b) Find the surface area of the side of this cone for $0 \leq y \leq 3$.

Sol'n: 1) General formula from topic 2 ☐:

$$A = \iint_S |\vec{r}_u \times \vec{r}_v| \, du \, dv,$$

where $\vec{r}(u, v)$ is given a few lines above.

2) Determine the limits for S :

$$0 \leq y \leq 3 \Rightarrow 0 \leq 2u \leq 3 \Rightarrow 0 \leq u \leq 3/2;$$

the cone is fully rotated (not by half a turn, etc.), $\Rightarrow 0 \leq v \leq 2\pi$.

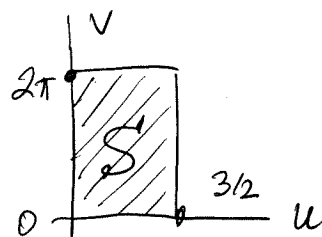
3) Compute $|\vec{r}_u \times \vec{r}_v|$.

$$\vec{r} = \langle u \cdot \cos v, 2u, u \cdot \sin v \rangle$$

$$\vec{r}_u = \langle \cos v, 2, \sin v \rangle$$

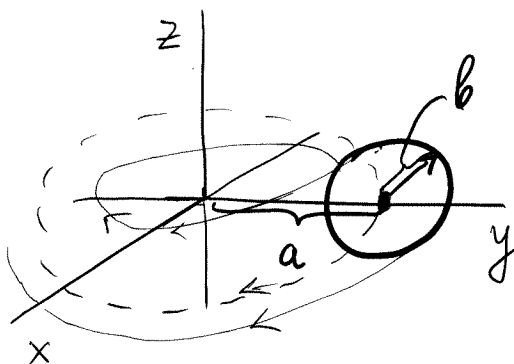
$$\vec{r}_v = \langle -u \sin v, 0, u \cos v \rangle$$

$$\Rightarrow |\vec{r}_u \times \vec{r}_v| = \sqrt{2^2 + 1} \cdot u$$



4) Sub into formula: $A_{\text{total}} = \int_0^{2\pi} \int_0^{3/2} \sqrt{5} \, u \, du \, dv = \frac{\sqrt{5}}{2} \cdot \left(\frac{3}{2}\right)^2 \cdot 2\pi$

Ex. 4(a) Find the parametric eqs. of a torus (the surface of a donut);
see the Figure for #64 after Sec. 16.6.



Sol'n:

1) This torus is obtained by rotating a circle shown on the left around the z -axis.

2) Param. eqs of the circle (radius = b ; center @ $(0, a, 0)$):

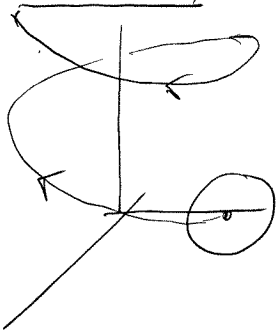
$$\begin{cases} y = a + b \cos u \\ z = b \sin u \end{cases}$$

3) Rotate about the z -axis:

$$\begin{cases} x = (a + b \cos u) \cdot \cos v \\ y = (a + b \cos u) \cdot \sin v \\ z = b \sin u \end{cases} \leftarrow \text{stays the same}$$

Param. eqs of a torus, explored in Lab 7.

Ex. 4(b) Find the param. eqs. of a spiraling tube:



See Fig. 4, or search online, or think of a "fat" spring (spiral).

1) This tube can be obtained by moving the center of the circle of Ex. 4(a) along a helix (the "must-see" curve from Sec. 13.1, Ex. 4). This is not a rotation, so the method described above needs a modification.

27-12

2) The "motion" in the xy -plane is still the same as for Ex. 4(a) — rotation (imagine looking at a spiral along its axis: you will see a circle).

However, along the z -axis it is some uniform change; so the eqs. of the spiraling tube are

$$\begin{aligned} x &= (a + b \cos u) \cdot \cos v \\ y &= (a + b \cos u) \cdot \sin v \\ z &= b \sin u + \underbrace{c \cdot v}_{\substack{\text{rate of} \\ \text{uniform } z\text{-change} \\ \text{of the center of the circle.}}} \end{aligned} \quad \begin{aligned} &\leftarrow \text{same as for the} \\ &\quad \text{torus} \\ &\leftarrow \text{this new term} \\ &\quad \text{describes the} \\ &\quad \text{uniform change} \end{aligned}$$

This object will be explored in Lab 7. //

⑤ Verifying the parametric equations of a surface

In Ex. 3(a) we derived the parametric eqs. of a cone. Here we will solve an "inverse" problem:

Ex. 5 Verify that the parametric eqs.

$$\begin{cases} x = a \cdot u \\ y = b \cdot u \cdot \cos v \\ z = b \cdot u \cdot \sin v \end{cases} \quad \begin{aligned} 0 &\leq u \leq c \\ \frac{\pi}{2} &\leq v \leq \pi \end{aligned}$$

define a quarter of a half-cone. Find the length and the radius at the base of this cone.

Sol'n: 1) Verify that the eqs. define a cone. The rotation is in the yz -plane, $\Rightarrow$ cone's axis is along x , $\Rightarrow$ the general form of its equation must be:

$$x = k \cdot \sqrt{y^2 + z^2} \quad (*)$$

for some $k = \text{const}$. Substitute the given eqs. into the l.h.s. & r.h.s. of $(*)$ and verify $(*)$:

$$\text{l.h.s. of } (*) = a \cdot u$$

$$\begin{aligned} \text{r.h.s. of } (*) &= k \cdot \sqrt{(bu \cdot \cos v)^2 + (bu \cdot \sin v)^2} \\ &= k \cdot \sqrt{(bu)^2 (\underbrace{\cos^2 v + \sin^2 v}_1)} = k \cdot bu \end{aligned}$$

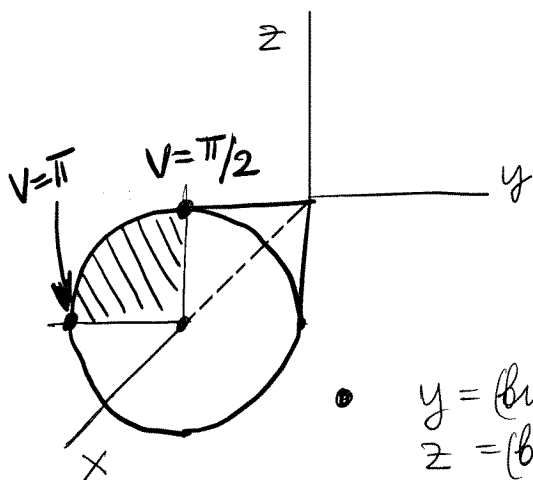
$$\text{l.h.s.} \stackrel{?}{=} \text{r.h.s.} \Rightarrow a \cdot u = k \cdot b \cdot u \Rightarrow k = \frac{a}{b}$$

Thus, this cone has the Cartesian equation

$$x = \frac{a}{b} \sqrt{y^2 + z^2}$$

2) Impose the given restrictions on the cone.

SKETCH:



• length:

$$0 \leq u \leq c$$

$$0 \leq \underline{a \cdot u} \leq ac$$

$$0 \leq x \leq \underline{ac}$$

• radius at base:

$$y = b \cdot u_{\max} = \underline{bc}$$

$$\begin{aligned} y &= (bu) \cdot \cos v & \frac{\pi}{2} \leq v \leq \pi \\ z &= (bu) \cdot \sin v \end{aligned}$$

$$\text{@ } v = \pi/2: (y=0, z>0) \quad \text{@ } v = \pi: (y<0, z=0)$$

Thus, the dashed part of the cone is what we need.

