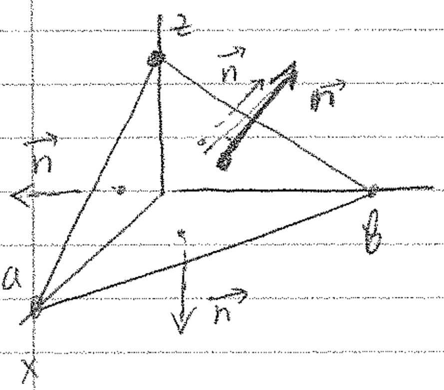
Two extra examples (if time permits)
(do one example per class;
it's better to do 1 example in 1 class that 2 examples,
if one has only one class left).

Extra Ex. 1 (#10, Sec. 16.9)
Find the flux through the surface of the pyramid bounded by the coord. planes and plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. $\vec{F}=\langle z, y, zx\rangle$.

Sol'n:
0) It's inconvenient to compute flux through 4 sides of the pyramid, using 4 different $\vec{n}$'s.
So use Gauss Thm instead.

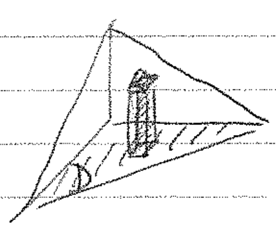


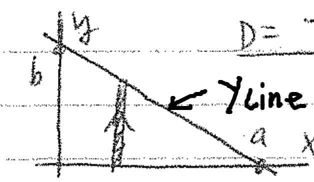
1) Sketch.
Explain the intersepts of the plane with coord. axes.

2) $$\oiint_S \vec{F}\circ\vec{n}\,dS \underset{\substack{\uparrow\\ \text{Gauss}}}{=} \iiint_E \operatorname{div}\vec{F}\,dV.$$

$$\operatorname{div}\vec{F}=\frac{\partial z}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial(zx)}{\partial z}=0+1+x=1+x.$$

3) $$\iiint_E \dots dV=\underbrace{\iint_D\left(\int_{z_{bot}=0}^{z_{top}=z_{plane}=c(1-\frac{x}{a}-\frac{y}{b})}\dots\,dz\right)dxdy}_{\text{discuss this as a reminder}}$$



4)


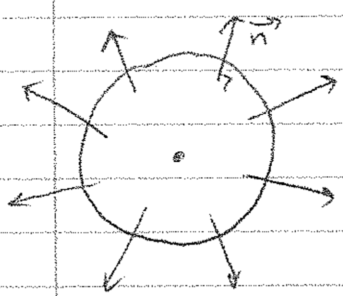
D = Type I: $0\le y\le y_{line}$
$0\le x\le a$

Eq. of line: $\overset{b}{y_0}$ in $y-\overset{b}{y_0} = \overset{\text{slope, } = -\frac{b}{a}}{m}(x-\overset{0}{x_0})$, $\Rightarrow$

$y = -\frac{b}{a}x + b$.

Thus: $\text{Flux} = \int_0^a \int_0^{(-b/a)x+b} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} (1+x)\,dz\,dy\,dx$.

Extra Ex. 2 Find the flux of field $\vec{F} = |\vec{r}|^2 \cdot \vec{r}$ through the sphere of radius R, centered @ $(0,0,0)$. Do this by Gauss Thm, and then directly.



Sol'n: First sketch. Then state

$$\oiint_S \vec{F}\cdot\vec{n}\,dS = \iiint_E \operatorname{div}\vec{F}\,dV.$$

Observe that $\operatorname{div}\vec{F} \neq 0$ ($\vec{F}$ diverges), $\Rightarrow$ flux must be $\neq 0$.

A) By Gauss Thm.

1) $\operatorname{div}\vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \bullet \Big[\overbrace{(x^2+y^2+z^2)}^{|\vec{r}|^2}\langle x,y,z\rangle\Big]$

$= \frac{\partial}{\partial x}\big((x^2+y^2+z^2)x\big) + \underset{y\ \&\ z}{\text{similar for}} = \big((x^2+y^2+z^2)\cdot 1 + 2x\cdot x\big) +$

$+\big((x^2+y^2+z^2)+2y^2\big) + \big((x^2+y^2+z^2)+2z^2\big) = (3+2)(x^2+y^2+z^2)$

$= 5\rho^2.$

2) Do the integral in spherical coord's.

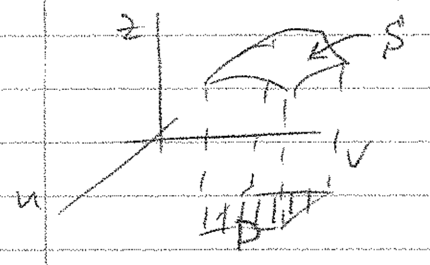
$$\iiint_E \operatorname{div}\vec{F}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} 5\rho^2\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Note: $\int_0^R 5\rho^4\,d\rho = R^5$. (will compare with B) below)

B) <u>By Direct Calculation</u>

Recall from Sec. 16.7: $\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_D (\vec{F}\cdot\vec{n})\cdot|\vec{r}_u\times\vec{r}_v|\,du\,dv$

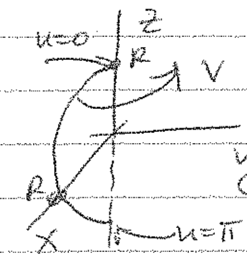
where D is the "projection" of S into uv-plane.

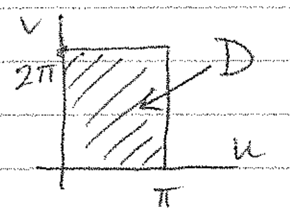


So we need: D, $\vec{r}$, $\vec{r}_u\times\vec{r}_v$.

1) By Ex. 2 of Notes 16.6:

$x = R\sin u\cdot\cos v$
$y = (R\sin u)\cdot\sin v$
$z = R\cos u$





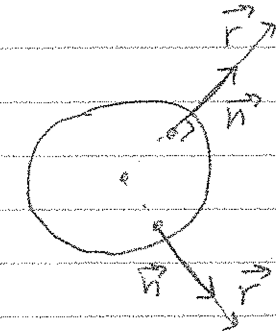
$\vec{r} = \langle R\sin u\cos v,\ R\sin u\cdot\sin v,\ R\cos u\rangle$

2) $\vec{r}_u = \langle R\cos u\cos v,\ R\cos u\sin v,\ -R\sin u\rangle$

$\vec{r}_v = \langle -R\sin u\cdot\sin v,\ R\sin u\cos v,\ 0\rangle$.

$\vec{r}_u\times\vec{r}_v = R\cdot\sin u\cdot\vec{r}$

calculation



3) $\vec{n} = \pm\dfrac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}$.

Since at every point on sphere, $\vec{n}\uparrow\uparrow\vec{r}$, and we know:

$\vec{r}_u\times\vec{r}_v = (\text{positive \#})\cdot\vec{r}$,

$\Rightarrow \vec{n} = + \dfrac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}$.

Then

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_D \vec{F}\cdot\frac{\vec{r}_u\times\vec{r}_v}{\cancel{|\vec{r}_u\times\vec{r}_v|}}\cdot\cancel{|\vec{r}_u\times\vec{r}_v|}\,du\,dv = \iint_D \underbrace{|\vec{r}|^2}_{R^2 \text{ on sphere}}\cdot\vec{r}\cdot R\sin u\,\vec{r}\,du\,dv$$

$$= \int_0^{2\pi}\int_0^{\pi} R^3\cdot\sin u\cdot\underbrace{(\vec{r}\cdot\vec{r})}_{|\vec{r}|^2 = R^2 \text{ on sphere}}\,du\,dv = \int_0^{2\pi}\int_0^{\pi} R^5\sin u\cdot du\,dv.$$

Same as in A).